English

Contact during the exam:
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Allowed help: C: Simple calculator allowed. Handbook of mathematical formulas allowed.

EXAM IN FE 8100 QUANTUM COMPUTATION AND QUANTUM COMMUNICATION

Thursday 9. December 2010
Time: 09:00 – 13:00

Problem 1

a) Consider a qubit eraser, i.e., a black box with one input and one output, transforming any qubit $|\psi\rangle$ to the fixed standard state $|0\rangle$. Prove that the qubit eraser cannot be realized as a single qubit unitary transformation.

b) We now append an extra qubit $|0\rangle$ to the input, such that the total input state is $|\psi\rangle|0\rangle$. Construct a unitary operation on the two qubits such that the first qubit transforms as $|\psi\rangle \to |0\rangle$ for any $|\psi\rangle$.

c) The transformation of the qubit eraser can be written $\rho \to \mathcal{E}(\rho)$, where ρ is the input state (density operator), and $\mathcal{E}(\rho)$ is a quantum operation. Write $\mathcal{E}(\rho)$ in the operator-sum formalism. (That is, write $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ and find the E_k's.)

Problem 2

a) The composite system AB is in the state $|\psi\rangle = Ce^{i\theta}\left(|00\rangle + |{+}{+}\rangle\right)$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and $C > 0$. (The two parts of the state correspond to subsystem A and B, respectively.) Calculate C. Is it possible to construct a measurement to read out the parameter θ if you have access to several copies of $|\psi\rangle$? Why/why not?

b) What is the state of system A, ρ_A?

c) Is the state ρ_A mixed or pure? Is there any entanglement between A and B? Give the reason.

d) Based on the outcome of flipping a coin, Charlie prepares either the state

$$\sigma_0 = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

or

$$\sigma_1 = |0\rangle\langle 0|.$$

He gives his state (but not the outcome of the coin) to Dolly. By optimizing her measurement, Dolly now tries to find out the outcome of Charlies coin. Calculate an upper bound on the mutual information between her result and the outcome of Charlies coin.

Problem 3

Perfect photonic amplifiers do not exist. Amplifiers either generate noise / operate imperfectly, or they operate nondeterministically. Here we will consider the latter possibility.

a) Consider the setup in Fig. 1. The input state is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Here $|0\rangle$ is the vacuum state (no photons), while $|1\rangle$ is the single photon state. Show that the total state before the measurements is

$$\begin{aligned} |\phi\rangle &= \alpha\sqrt{\eta}|001\rangle + \alpha\frac{\sqrt{1-\eta}}{\sqrt{2}}|010\rangle + \alpha\frac{\sqrt{1-\eta}}{\sqrt{2}}|100\rangle \\ &\quad - \beta\frac{\sqrt{\eta}}{\sqrt{2}}|101\rangle + \beta\frac{\sqrt{\eta}}{\sqrt{2}}|011\rangle - \beta\frac{\sqrt{1-\eta}}{\sqrt{2}}|200\rangle + \beta\frac{\sqrt{1-\eta}}{\sqrt{2}}|020\rangle. \end{aligned} \tag{1}$$

Here the state $|ijk\rangle$ means i photons in the mode going to detector A, j photons in the mode going to detector B, and k photons in the output mode.

If you are not familiar with photonics and linear optics, you can still solve the next problems, taking Eq. (1) for granted, and noting that the basis states above are orthonormal, $\langle ijk|i'j'k'\rangle = \delta_{ii'}\delta_{jj'}\delta_{kk'}$.

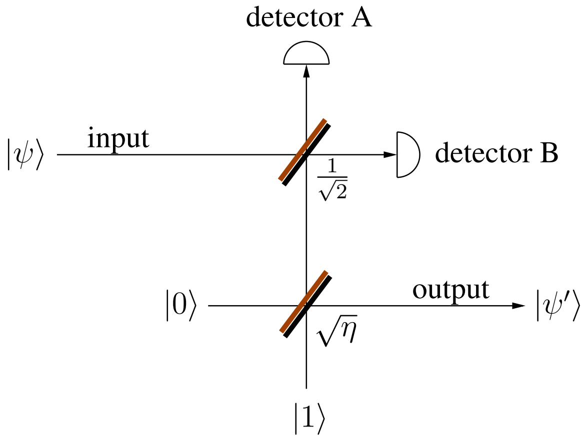


Figure 1: A nondeterministic amplifier (Ralph and Lund, arXiv:0809.0326, 2008). For the lower beam splitter, the reflection coefficient from the black side is $\sqrt{\eta}$, from the other side $-\sqrt{\eta}$, and the transmission coefficients are $\sqrt{1-\eta}$. The same sign convention applies to the upper beam splitter; however here the reflection and transmission coefficients are $\pm\frac{1}{\sqrt{2}}$.

b) What is the probability p of detecting no photons in detector A and a single photon in detector B?

c) If no photons are detected in detector A, and a single photon in detector B, what is the post-measurement state? Write the *normalized* post-measurement state in the form

$$|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle, \tag{2}$$

and calculate the amplification $|\beta'/\beta|$.

d) The nondeterministic, perfect amplifier is apparently successful when no photons have been detected in A and a single photon in B. By a small modification to the setup, how can the probability of successful amplification be increased from p to $2p$? (Hint: Consider what happens when a single photon is detected in A and no photons in B.) Sketch the amplification $|\beta'/\beta|$ and the probability of success $2p$ roughly as a function of η when $|\beta|^2 \ll 1-\eta$.

SOME FORMULAS

Single qubit operations:

$$X = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad Z = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$R_x(\theta) \equiv e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_y(\theta) \equiv e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_z(\theta) \equiv e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

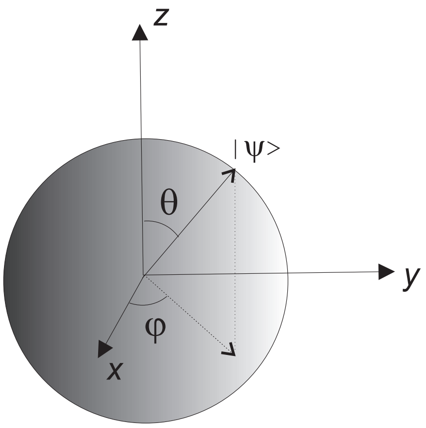
Algebra for the Pauli matrices $\sigma_1 = X$, $\sigma_2 = Y$, and $\sigma_3 = Z$:

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \quad [\sigma_2, \sigma_3] = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2.$$
$$\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i = 2I\delta_{ij} \quad \text{for} \quad i, j = 1, 2, 3.$$

Bloch sphere representation:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$
$$a = \cos(\theta/2), \quad b = e^{i\varphi}\sin(\theta/2)$$

The state $|\psi\rangle$ has the angular coordinates (θ, ϕ): R_x, R_y, and R_z are rotations around the x, y, and z axes on the Bloch sphere.



The Holevo bound: You are given a state ρ_X, where $X = 0, \ldots, n$ with associated probabilities $p_0, \ldots, p_n$. For any measurement you can do (with result Y), $H(X:Y) \leq S(\rho) - \sum_{x=1}^{n} p_x S(\rho_x), \quad \rho = \sum_{x=1}^{n} p_x \rho_x$.

von Neumann entropy: $S(\rho) = -\mathrm{Tr}\{\rho \log(\rho)\}$.

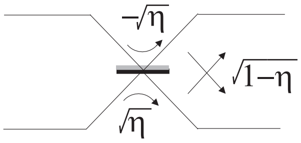
Fidelity: $F(\rho, \sigma) = \mathrm{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}$

Trace Distance: $D(\rho, \sigma) = \frac{1}{2}\mathrm{Tr}\,|\rho - \sigma|$

Functions of normal matrices: $fA = \sum_i f(\lambda_i)\,|i\rangle\,\langle i|$

Projective maesurements: $p(m) = \mathrm{Tr}(P_m \rho)$. Projectors are Hermitian and satisfy $P_m^2 = P_m$.

Beamsplitter: The reflection cofficient from the grey side is $-\sqrt{\eta}$; the reflection coefficient from the black side is $\sqrt{\eta}$; and both transmission coefficients are $\sqrt{1-\eta}$.



Number states:

$$
\begin{aligned}
a|n\rangle &= \sqrt{n}|n-1\rangle \\
a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\
\langle m|n\rangle &= \delta_{mn}, \\
|n\rangle &= \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.
\end{aligned}
$$

Linear optics transformations:

Heisenberg picture:

$$a_i \rightarrow b_i = \sum_{j=1}^{N} S_{ij} a_j,$$

Schrodinger picture:

1. The state $|n_N, ..., n_2, n_1\rangle$ becomes a superposition of terms on the form $|m_N, ..., m_2, m_1\rangle$, where the total number of photons is conserved,
   $m_1 + m_2 + ... + m_N = n_1 + n_2 + ... + n_N$.
2. The probability amplitude for the transition $|n_N, ..., n_2, n_1\rangle \rightarrow |m_N, ..., m_2, m_1\rangle$ is

   $$\sum_{\text{paths}} C \cdot S_{\text{path photon 1}} \cdots S_{\text{path photon } n_1 + n_2 + ... + n_N},$$

   where $C = \sqrt{\frac{m_1! m_2! \cdots m_N!}{n_1! n_2! \cdots n_N!}}$.
   The terms in the sum represent the possible paths taken by the photons.