Solution
FE 8100 December 2010

Problem 1

a) Assuming U does the job, we have $U|\psi\rangle = |0\rangle$ and $U|\phi\rangle = |0\rangle$ for any $|\psi\rangle$ and $|\phi\rangle$. Taking the inner product of these two equations we find that $\langle\phi|\psi\rangle = 1$, which contradicts the fact that $|\psi\rangle$ and $|\phi\rangle$ were supposed to be arbitrary.

b) We let the two qubits transform as follows: $|\psi\rangle|0\rangle \to |0\rangle|\psi\rangle$. Clearly, this is a unitary operation; the operation is obtained simply by exchanging the two qubits.

c) For example,

$$\mathcal{E}(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |0\rangle\langle 1|\rho|1\rangle\langle 0|,$$

with operator elements $|0\rangle\langle 0|$ and $|0\rangle\langle 1|$. There are several other set of operator elements due to the unitary freedom in the operator-sum formalism.

Problem 2

a) By calculating the inner product $\langle\psi|\psi\rangle$, we find $\langle\psi|\psi\rangle = 3C^2$. Thus $C = 1/\sqrt{3}$ since the state must be normalized. For any measurement operator M_m, the corresponding probability is $\langle\psi|M_m^\dagger M_m|\psi\rangle = C^2\left(\langle 00| + \langle{+}{+}|\right)|M_m^\dagger M_m\left(|00\rangle + |{+}{+}\rangle\right)$. Since the probability is independent of θ, there is no way to read out θ.

b) The density operator of AB is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{3}\left(|00\rangle\langle 00| + |00\rangle\langle{+}{+}| + |{+}{+}\rangle\langle 00| + |{+}{+}\rangle\langle{+}{+}|\right).$$

Tracing out system B we get

$$\rho_A = \mathrm{Tr}_B(\rho) = \frac{1}{3}\left(|0\rangle\langle 0| + \frac{1}{\sqrt{2}}|0\rangle\langle +| + \frac{1}{\sqrt{2}}|+\rangle\langle 0| + |+\rangle\langle +|\right)$$
$$= \frac{1}{6}\left(5|0\rangle\langle 0| + 2|0\rangle\langle 1| + 2|1\rangle\langle 0| + |1\rangle\langle 1|\right).$$

c) We first check whether ρ_A is a mixed state. Calculating ρ_A^2, we find that $\mathrm{Tr}(\rho_A^2) = 17/18 < 1$. Thus the eigenvalues of ρ_A are not 0 or 1, but some values in between. It follows that ρ_A is mixed. Since the total state ρ is pure, while ρ_A is mixed, systems A and B must be entangled.

d) The upper bound is given by Holevos bound, $H(C:D) \leq S(\sigma) - \frac{1}{2}S(\sigma_0) - \frac{1}{2}S(\sigma_1)$, where $\sigma = \frac{1}{2}(\sigma_0 + \sigma_1)$. We obtain

$$H(C:D) \leq S(\sigma) - 1/2 - 0 = h(1/4) - 1/2 = \frac{3}{4}(2 - \log 3),$$

where $h(\cdot)$ is the binary Shannon entropy function.

Problem 3

a) We apply the following input-output relations for a beam splitter:

$$\begin{aligned}
&|00\rangle \rightarrow |00\rangle \\
&|01\rangle \rightarrow \sqrt{\eta}|01\rangle + \sqrt{1-\eta}|10\rangle \\
&|10\rangle \rightarrow -\sqrt{\eta}|10\rangle + \sqrt{1-\eta}|01\rangle \\
&|11\rangle \rightarrow (\sqrt{1-\eta}\sqrt{1-\eta} - \sqrt{\eta}\sqrt{\eta})|11\rangle - \sqrt{2}\sqrt{\eta}\sqrt{1-\eta}|20\rangle + \sqrt{2}\sqrt{\eta}\sqrt{1-\eta}|02\rangle \\
&\quad = (1-2\eta)|11\rangle - \sqrt{2\eta}\sqrt{1-\eta}|20\rangle + \sqrt{2\eta}\sqrt{1-\eta}|02\rangle
\end{aligned}$$

The initial state is

$$|\psi\rangle|0\rangle|1\rangle = \alpha|001\rangle + \beta|101\rangle.$$

Applying the lower beam splitter, we obtain

$$\alpha\sqrt{\eta}|001\rangle + \alpha\sqrt{1-\eta}|010\rangle + \beta\sqrt{\eta}|101\rangle + \beta\sqrt{1-\eta}|110\rangle.$$

Applying the upper beam splitter, we obtain

$$\begin{aligned}
|\phi\rangle = {}& \alpha\sqrt{\eta}|001\rangle + \alpha\frac{\sqrt{1-\eta}}{\sqrt{2}}|010\rangle + \alpha\frac{\sqrt{1-\eta}}{\sqrt{2}}|100\rangle \\
&- \beta\frac{\sqrt{\eta}}{\sqrt{2}}|101\rangle + \beta\frac{\sqrt{\eta}}{\sqrt{2}}|011\rangle - \beta\frac{\sqrt{1-\eta}}{\sqrt{2}}|200\rangle + \beta\frac{\sqrt{1-\eta}}{\sqrt{2}}|020\rangle. \qquad (1)
\end{aligned}$$

b)

$$p = \frac{|\alpha|^2(1-\eta) + |\beta|^2\eta}{2} = \frac{|\beta|^2(2\eta-1) + 1 - \eta}{2}.$$

c)

$$|\psi'\rangle = \frac{1}{\sqrt{p}}\left(\alpha\frac{\sqrt{1-\eta}}{\sqrt{2}}|0\rangle + \beta\frac{\sqrt{\eta}}{\sqrt{2}}|1\rangle\right).$$

Amplification $|\beta'/\beta| = \sqrt{\frac{\eta}{2p}} = \sqrt{\frac{\eta}{|\beta|^2(2\eta-1)+1-\eta}}$.

d) When a single photon is detected in A and no photons in B, we get the same postmeasurement state as above; however the sign of the $|1\rangle$ term is opposite. This can be fixed by a Z-operation conditional on the measurement results. (If no photons in A and a single photon in B, then nothing is applied; when a single photon is detected in A and none in B, then a phase shift Z operation is applied to the output.)

When $|\beta|^2 \ll 1-\eta$ we can approximate $2p = 1-\eta$. The amplification becomes $\sqrt{\frac{\eta}{1-\eta}}$. Thus the amplification is larger than unity for $\eta > 1/2$. As η increases, the amplification increases but the probability of success decreases. Within our approximation, as $\eta \to 1$ the amplification tends to infinity while the probability tends to zero. (However, note that the approximation $|\beta|^2 \ll 1-\eta$ breaks down in this limit unless simultaneously $|\beta|^2 \to 0$!)