# Solution Manual Exercise 7
# Quantum Circuits

## 8.4

We use the computational basis as the basis for the environment. The operation elements are then,

$$E_0 = \langle 0| (P_0 \otimes I + P_1 \otimes X) |0\rangle = P_0$$
$$E_1 = \langle 0| (P_0 \otimes I + P_1 \otimes X) |1\rangle = P_1.$$

## 8.5

$$E_0 = \frac{1}{\sqrt{2}}X; \qquad E_1 = \frac{1}{\sqrt{2}}Y$$

## 8.15

Using $\rho = \frac{I+\vec{r}\cdot\vec{\sigma}}{2}$ we get

$$\mathcal{E}(\rho) = \frac{1}{2}(I + r_x) |+\rangle \langle +| + \frac{1}{2}(I - r_x) |-\rangle \langle -| = I + r_x X$$

Thus this measurement projects the Bloch sphere onto its x-axis. Note that this is the same operation as doing a bit flip with probability $p = 0.5$.

## 8.17

Using the commutators $[\sigma_i \sigma_j] = \delta_{ij}$ it is straightforward to verify 8.105. Using the Bloch representation $\rho = \frac{I+\vec{r}\cdot\vec{\sigma}}{2}$ and that quantum operations are linear transformations on density matrices (8.43) we get

$$\mathcal{E}(\rho) = \frac{1}{2}\mathcal{E}(I) + \sum_i \frac{r_i}{2}\mathcal{E}(\sigma_i) = \frac{I}{2}$$

## 8.26

We first assume that $\rho_{\text{in}}$ is a pure state. The inital state of the enitire system is then $|\Psi_{in}\rangle = a|00\rangle + b|10\rangle$. Appling the controlled rotation operator we get

$$|\Psi_R\rangle = a|00\rangle + b|1\rangle R_y(\theta)|0\rangle = a|00\rangle + b\cos\frac{\theta}{2}|10\rangle + b\sin\frac{\theta}{2}|11\rangle .$$

After having measured the environment in the computational basis (without knowing the result) the stystem is now in a mixed state,

$$\begin{aligned}\rho_M &= \sum_m M_m|\Psi_R\rangle\langle\Psi_R| M_m^\dagger \\ &= \left(|a|^2|0\rangle\langle 0| + ab^*\cos\frac{\theta}{2}|0\rangle\langle 1| + a^*b\cos\frac{\theta}{2}|1\rangle\langle 0| + |b|^2\cos^2\frac{\theta}{2}|1\rangle\langle 1|\right)\otimes|0\rangle\langle 0| \\ &\quad + |b|^2\sin^2\frac{\theta}{2}|1\rangle\langle 1|\otimes|1\rangle\langle 1|\end{aligned}$$

Tracing away the environment we get

$$\rho_{\text{out}} = |a|^2|0\rangle\langle 0| + ab^*\cos\frac{\theta}{2}|0\rangle\langle 1| + a^*b\cos\frac{\theta}{2}|1\rangle\langle 0| + |b|^2|1\rangle\langle 1| .$$

This is phase damping with $\sin\frac{\theta}{2} = \sqrt{1-\lambda}$.

If the initail state is a mixed state, $\rho_{\text{in}} = \sum_i p_i\rho_i$ with $\rho_i = |\Psi_i\rangle\langle\Psi_i|$, each state $\rho_i$ will be subject to phase damping as described above. The transformation of the mixed state is then

$$\rho_{\text{in}} = \sum_i p_i\begin{pmatrix}|a_i|^2 & a_ib_i^* \\ a_i^*b_i & |b_i|^2\end{pmatrix} \xrightarrow{\mathcal{E}} \sum_i p_i\begin{pmatrix}|a_i|^2 & a_ib_i^*\cos\frac{\theta}{2} \\ a_i^*b_i\cos\frac{\theta}{2} & |b_i|^2\end{pmatrix}$$

We see that mixed states are subject to the same phase damping as pure states.