

THE OPTIMAL WEIGHTS OF A MAXIMUM RATIO COMBINER USING AN EIGENFILTER APPROACH

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ABSTRACT

This letter presents an eigenfilter approach to obtain the optimal weights of a diversity receiver using maximum ratio combining (MRC). It is shown that the optimal weights correspond to a matched filter solution for stochastic signals and that the optimal weights are the same regardless of the signal branch correlation.

1. INTRODUCTION

Maximum ratio combining (MRC) is the optimum spatial diversity strategy to reduce the signal fluctuations caused by multipath propagation in wireless communications. By definition, a MRC combiner linearly combines the individually received branch signals so as to maximize the *instantaneous* output signal-to-noise ratio (SNR). In a recent letter by Dong and Beaulieu [1], it was proved that a MRC combiner operating on correlated branches is optimal even if the branch signals are weighted as though they were independent. A decorrelation approach was used to show that the expressions for the signal envelope and the SNR were the same regardless of the signal branch correlation. In this letter, it is shown that there is actually no need to look at the correlation between the branches at all to establish this statement. The optimal weights of a MRC combiner is optimal (in a signal-to-noise ratio sense) regardless of the signal branch correlation due to the optimization of the instantaneous SNR instead of the average SNR. The optimal weights originally derived in [2] by using the Schwarz inequality are obtained in this letter by means of an eigenfilter approach.

2. CHANNEL MODEL

In general, the wireless channel between n_T transmit antennas and n_R receive antennas can be represented

by a channel matrix \mathbf{H} written as

$$\mathbf{H} = \begin{bmatrix} h_{11} & \cdots & h_{1n_T} \\ h_{21} & \cdots & h_{2n_T} \\ \vdots & \ddots & \vdots \\ h_{n_R1} & \cdots & h_{n_Rn_T} \end{bmatrix}. \quad (1)$$

For a narrowband flat-fading channel, a general entry of the channel matrix is denoted by

$$\begin{aligned} h_{ij} &= \alpha + j\beta \\ &= \sqrt{\alpha^2 + \beta^2} \cdot e^{-j \arctan \frac{\beta}{\alpha}} \\ &= |h_{ij}| \cdot e^{j\phi_{ij}}, \end{aligned} \quad (2)$$

where $|h_{ij}|$ represents the fading envelope of the narrowband channel between the j th transmitter and the i th receiver. If both α and β are zero-mean, independent, and normally distributed random variables, h_{ij} is a zero-mean, complex, normally distributed random variable with a fading envelope $|h_{ij}|$ following a Rayleigh distribution. For non-line-of-sight (NLOS) transmission, it is common to assume all the entries of the channel matrix as (possibly correlated) complex random variables with Rayleigh distributed envelopes [3]. For the special case of $n_T = 1$, the channel matrix is reduced to a column vector. Denoting the transmitted (stochastic) complex baseband signal at time index $k \in \{0, 1, 2, \dots\}$ by $x(k)$, the received signal at antenna $i \in \{1, 2, \dots, n_R\}$ can be written as

$$s_i(k) = h_i(k)x(k) + n_i(k). \quad (3)$$

In (3), $h_i(k)$ represents the time-varying complex channel gain between a single transmit antenna and the i th receive antenna at time index k . The complex-valued additive white gaussian noise (AWGN) is denoted $n_i(k)$ and is modelled as a complex zero-mean normally distributed random variable. It is assumed that $n_i(k)$ is independent of $h_i(k)$. Using vector notation, the received signal vector at time instant k may be expressed as

$$\mathbf{s}(k) = \mathbf{h}(k)x(k) + \mathbf{n}(k), \quad (4)$$

where $\mathbf{s}(k)$, $\mathbf{h}(k)$, and $\mathbf{n}(k)$ are all column vectors. For notational convenience, data associated with time k will from now on be indicated by subscripts, such as \mathbf{s}_k , \mathbf{h}_k , and \mathbf{n}_k . In the next section, an eigenfilter approach is used to obtain the optimal linear combination of the entries of \mathbf{s}_k in order to maximize the output SNR.

3. EIGENFILTER APPROACH

With a linear combining technique, the output of the maximum ratio combiner at time instant k may be expressed as

$$\begin{aligned} y_k &= \mathbf{w}^H \mathbf{s}_k \\ &= \mathbf{w}^H \mathbf{h}_k x_k + \mathbf{w}^H \mathbf{n}_k, \end{aligned} \quad (5)$$

where $\mathbf{w} \in \mathbb{C}^{n_R}$ are the (as yet unknown) weights of the linear combiner and the superscript H denotes Hermitian transpose. In a fading channel, the received SNR level will vary due to random fluctuation of the signal power. This random fluctuation is described by the fading envelope $|h_{ij}|$. In this letter, the fading envelopes associated with the separate antenna branches are represented by the entries of the channel vector \mathbf{h} . In general, the channel vector \mathbf{h} is treated as a multivariate random variable with possibly correlated entries. However, the current (observed) channel vector at time instant k is always treated as a single channel realization of the random variable \mathbf{h} . Invoking this assumption together with the assumption of an average transmit power P_T , the instantaneous received *signal* power S at the output of the MRC combiner at time instant k can be expressed as

$$S = P_T \cdot \mathbf{w}^H \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}. \quad (7)$$

The expected output *noise* power N may be expressed as

$$N = \mathcal{E}\{|\mathbf{w}^H \mathbf{n}|^2\} = \mathbf{w}^H \mathbf{R}_n \mathbf{w}, \quad (8)$$

where $\mathbf{R}_n = \mathcal{E}\{\mathbf{n}\mathbf{n}^H\}$. Using (7) and (8), the instantaneous output SNR from the MRC at time instant k can be expressed as

$$\gamma_{k,MRC} = \frac{S}{N} = \frac{P_T \cdot \mathbf{w}^H \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}}{\mathbf{w}^H \mathbf{R}_n \mathbf{w}} = \frac{P_T \cdot \mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w}}{\mathbf{w}^H \mathbf{R}_n \mathbf{w}}, \quad (9)$$

where $\hat{\mathbf{R}}_k = \mathbf{h}_k \mathbf{h}_k^H$. Note that the numerator in (9) represents the output signal power for a single observed channel realization \mathbf{h}_k of the random variable \mathbf{h} at time instant k . The optimization problem is now to determine the coefficient vector \mathbf{w} so as to maximize the instantaneous output SNR for the current observed channel vector \mathbf{h}_k . In the next two subsections, the optimum weight vector solution for both equal and unequal branch noise powers will be derived.

3.1. Equal noise power

In the following, we assume equal noise power at each of the branches, and also that the noise is uncorrelated between the branches. This means that the noise correlation matrix introduced in the previous section can be represented as $\mathbf{R}_n = \sigma^2 \mathbf{I}$, where σ^2 represents the noise power common to all branches and \mathbf{I} represents the identity matrix. Taking the *conjugate derivative* $\partial/\partial \mathbf{w}^*$ [4] of (9) with respect to the weight vector \mathbf{w} , we obtain the following set of implications:

$$\begin{aligned} \frac{\partial \gamma_{k,MRC}}{\partial \mathbf{w}^*} &= 0 \\ &\Downarrow \\ P_T \cdot \hat{\mathbf{R}}_k \mathbf{w} (\sigma^2 \mathbf{w}^H \mathbf{w}) &= \sigma^2 \mathbf{w} (P_T \cdot \mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w}) \\ &\Downarrow \\ \hat{\mathbf{R}}_k \mathbf{w} &= \left(\frac{\mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w}}{\mathbf{w}^H \mathbf{w}} \right) \mathbf{w}. \end{aligned} \quad (10)$$

Introducing $\lambda = \frac{\mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w}}{\mathbf{w}^H \mathbf{w}}$ and using the result of (10) in (9), we obtain

$$\gamma_{k,MRC} = \frac{P_T \cdot \mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}} = \frac{P_T \cdot \mathbf{w}^H \lambda \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}} = \frac{P_T \cdot \lambda}{\sigma^2}. \quad (11)$$

It can be seen that the maximum value of the output SNR is given as $\frac{P_T \cdot \lambda_{max}}{\sigma^2}$, where λ_{max} is the largest eigenvalue of the matrix $\hat{\mathbf{R}}_k$. The optimal weight vector that yields the maximum output SNR is given by the eigenvector associated with λ_{max} . To obtain a non-trivial solution of (10), \mathbf{w} must reside within the column space (range) of $\hat{\mathbf{R}}_k$, denoted $\mathcal{R}(\hat{\mathbf{R}}_k)$. Since all the columns in $\hat{\mathbf{R}}_k$ are linear combinations of the single vector \mathbf{h}_k , it is a rank one matrix and $\mathcal{R}(\hat{\mathbf{R}}_k) = \{\mathbf{h}_k\}$. Since $\mathcal{R}(\hat{\mathbf{R}}_k)$ consists of just a single vector, the only way of obtaining a non-trivial solution is to select $\mathbf{w}_{opt} = c \cdot \mathbf{h}_k$ for an arbitrary $c \neq 0$. Inserting this result into the left side of (10), we obtain

$$\|\mathbf{h}_k\|^2 c \cdot \mathbf{h}_k = \lambda_{max} \mathbf{w}_{opt}. \quad (12)$$

The maximum eigenvalue is thus identified as $\lambda_{max} = \|\mathbf{h}_k\|^2$. Using this result in (11), the maximum output SNR is given by

$$\gamma_{k,MRC}^{max} = \frac{P_T \cdot \|\mathbf{h}_k\|^2}{\sigma^2} = \sum_{i=1}^{n_R} \frac{P_T \cdot |h_{k,i}|^2}{\sigma^2} = \sum_{i=1}^{n_R} \gamma_{k,i}, \quad (13)$$

where $|h_{k,i}|$ and $\gamma_{k,i}$ denote the fading envelope and the instantaneous SNR at the i th branch at time index k respectively. This result confirms that the output SNR may be expressed as a sum of the SNR values from the individual branches, which is a specific feature of the MRC combiner [2].

3.2. Unequal noise power

With unequal noise power in the diversity branches, the SNR expressed in (9) can not be simplified. Taking the conjugate derivative of (9) with respect to the weight vector \mathbf{w} , we obtain the following set of implications:

$$\begin{aligned} \frac{\partial \gamma_{k,MRC}}{\partial \mathbf{w}^*} &= 0 \\ &\Downarrow \\ P_T \cdot \hat{\mathbf{R}}_k \mathbf{w} (\mathbf{w}^H \mathbf{R}_n \mathbf{w}) &= (P_T \cdot \mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w}) \mathbf{R}_n \mathbf{w} \\ &\Downarrow \\ \hat{\mathbf{R}}_k \mathbf{w} &= \left(\frac{\mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w}}{\mathbf{w}^H \mathbf{R}_n \mathbf{w}} \right) \mathbf{R}_n \mathbf{w}. \end{aligned} \quad (14)$$

Introducing $\lambda = \frac{\mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w}}{\mathbf{w}^H \mathbf{R}_n \mathbf{w}}$ and exploit the fact that \mathbf{R}_n is nonsingular (diagonal matrix with all entries $\sigma_i^2 > 0$), (14) may be expressed as

$$\mathbf{R}_n^{-1} \hat{\mathbf{R}}_k \mathbf{w} = \lambda \mathbf{w}. \quad (15)$$

The optimal weight vector is now the eigenvector of the matrix $\mathbf{R}_n^{-1} \hat{\mathbf{R}}_k = \mathbf{R}_n^{-1} \mathbf{h}_k \mathbf{h}_k^H$ corresponding to the largest eigenvalue of the same matrix. By letting $\mathbf{h}'_k = \mathbf{R}_n^{-1} \mathbf{h}_k$, the current eigenvalue problem may be expressed as

$$\hat{\mathbf{R}}'_k \mathbf{w} = \lambda \mathbf{w}, \quad (16)$$

where $\hat{\mathbf{R}}'_k = \mathbf{h}'_k \mathbf{h}'_k{}^H$. A non-trivial solution of this equation is obtained if \mathbf{w} resides within the column space of $\hat{\mathbf{R}}'_k$, denoted $\mathcal{R}(\hat{\mathbf{R}}'_k)$. Every matrix of the simple form $\mathbf{h}'_k \mathbf{h}'_k{}^H$ has rank one [5] and the column space $\mathcal{R}(\hat{\mathbf{R}}'_k) = \{\mathbf{h}'_k\}$. As in the previous section, the only choice to obtain a non-trivial solution is to select $\mathbf{w}_{opt} = c \cdot \mathbf{h}'_k$ for an arbitrary $c \neq 0$. Inserting this result into the left side of (16), we obtain

$$\mathbf{h}_k^H \mathbf{R}_n^{-1} \mathbf{h}_k c \cdot \mathbf{h}'_k = \lambda_{max} \mathbf{w}_{opt}, \quad (17)$$

and the maximum eigenvalue can be identified as $\lambda_{max} = \mathbf{h}_k^H \mathbf{R}_n^{-1} \mathbf{h}_k$. Inserting the optimal weight vector $\mathbf{w}_{opt} = c \cdot \mathbf{h}'_k = c \cdot \mathbf{R}_n^{-1} \mathbf{h}_k$ into (9), we obtain

$$\gamma_{k,MRC}^{max} = P_T \cdot \mathbf{h}_k^H \mathbf{R}_n^{-1} \mathbf{h}_k = \sum_{i=1}^{n_R} \frac{P_T \cdot |h_{k,i}|^2}{\sigma_i^2} = \sum_{i=1}^{n_R} \gamma_{k,i}, \quad (18)$$

i.e., as in the previous subsection, the output SNR may be expressed as the sum of the SNR values from the individual branches. To summarize, we have obtained the following optimal weights:

$$\mathbf{w}_{opt} \propto \begin{cases} \mathbf{h}_k & \text{Equal noise power} \\ \mathbf{R}_n^{-1} \mathbf{h}_k & \text{Unequal noise power} \end{cases} \quad (19)$$

Using these optimal weights, the maximum output SNR from a MRC combiner may be expressed as

$$\gamma_{k,MRC}^{max} = \begin{cases} \frac{P_T \cdot \|\mathbf{h}_k\|^2}{\sigma^2} & \text{Equal noise power} \\ P_T \cdot \mathbf{h}_k^H \mathbf{R}_n^{-1} \mathbf{h}_k & \text{Unequal noise power} \end{cases} \quad (20)$$

Looking at the inner product in (5), the actual weights used in the inner product are the complex conjugate of the weights presented in (19). The optimal weights are then in agreement with the results presented in [2]. Since the solutions in this section is linked to an eigenvalue problem, the optimal (spatial) filter obtained using this method is commonly called an eigenfilter. As noted in [4], the optimum filter characterized in this way may be viewed as the stochastic counterpart of a matched filter.

4. INSTANTANEOUS VERSUS AVERAGE SNR

In [1], it is stated that the optimal performance of a MRC diversity receiver can be achieved without decorrelating the correlated input signals, but weighting the input branch signals as though they were independent. This statement seems to indicate (i) the possible existence of several optimal weight solutions depending on signal branch correlation (ii) that the optimal weights must be obtained by incorporating the correlation properties in the optimizing procedure. However, in this letter, the optimal weight vector is shown to be the same as in the original paper by Brennan [2] without taking the correlation properties of \mathbf{h} into account.

Regarding assertion (ii), the way to incorporate the correlation properties into the optimizing procedure is to optimize the average SNR instead of the instantaneous SNR. If such an approach had been pursued in this letter, the eigenvalue problem would have contained the correlation matrix $\mathbf{R}_h = \mathcal{E}[\mathbf{h}\mathbf{h}^H]$ instead of the (instantaneous) matrix $\hat{\mathbf{R}}_k = \mathbf{h}_k \mathbf{h}_k^H$. With the assumption of correlated branch signals however, \mathbf{R}_h would be a rank one matrix and the optimal weight vector solutions in (19) still apply. With the assumption of uncorrelated branch signals, \mathbf{R}_h is a diagonal (full rank) matrix and its eigenvalues are placed along its main diagonal. The optimal weight vector is still the eigenvector corresponding to the largest eigenvalue of the matrix \mathbf{R}_h . Assuming equal noise power in all branches for simplicity, the eigenvalues are equal to the individual channel powers. The solution is then to select the single channel realizing the largest channel gain. This selection scheme is commonly known as selection diversity and obviously, the results obtained by using average SNR are not in agreement with the results of MRC presented in [2].

Since the correlation properties of the channel vector \mathbf{h} in this paper have no influence on the optimality of the weight vector solution, nor is there any need to perform the decorrelation approach suggested in [1] in order to establish the fact that performance measures of MRC combining are identical for both correlated and uncorrelated branches. Even though the optimal solution for instantaneous SNR is the same regardless of the signal branch correlation, the actual performance of a MRC combiner on average will be highly dependent on the correlation properties.

5. CONCLUSION

An eigenfilter approach is used to obtain the optimal weights of a MRC combiner. At the same time, it is proved that the optimal solution is optimal regardless of the correlation between the diversity branches. This was recently established in [1] using a decorrelation approach.

6. REFERENCES

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