How Accurate are the Gaussian and Gamma Approximations to the Outage Capacity of MIMO Channels ? *

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Abstract— **Recent results showed that the capacity of a multiple-input-multiple-output (MIMO) fading channel can be approximated by a Gaussian random variable. This Gaussian approximation method has been used to approximate MIMO capacity complementary cumulative distribution functions, or equivalently, the outage capacity, for various fading scenarios where the exact moment generating functions (thus the moments) can be obtained. This paper studies and compares the accuracy of this Gaussian approximation with a newly proposed Gamma approximation for the cases in absence or presence of co-channel interference. It shows that while both approximations are very accurate even for a small number of antenna elements, various fading conditions, and a wide range of transmitting signal-to-noise ratio (TSNR), the MIMO capacity can be better approximated by a Gamma random variable for Rician channels especially for low to medium TSNR values.**

I. Introduction

Wireless communication systems employing multiple antenna elements at both the transmitter and the receiver have attracted much interest due to its significant capacity gain [1][2]. One important research direction has been to determine the capacity of such so-called "multiple-input-multiple-output" (MIMO) channels in a multipath fading environment. In this context, [3][4] proposed that the MIMO channel capacity can be accurately approximated by a Gaussian random variable, for the case when the channel state information (CSI) is not available at the transmitter but the receiver has the perfect channel state information. The implication of this Gaussian approximation is that only the capacity mean and variance are needed to get an accurate approximation to another important capacity measure: the capacity cumulative distribution function (CDF) (also known as the outage capacity), or equivalently, the capacity complementary cumulative distribution function (CCDF). The capacity variance of an independent and identically distributed (i.i.d.) MIMO Rayleigh channel was then derived in [3] and used to approximate the outage capacity, with the help of the capacity mean results obtained in [2]. More recently, the exact MIMO capacity moment generating function (MGF) results, and therefore the moments, when there is no co-channel interference were obtained for i.i.d. Rayleigh channels [4][5][6], independent but not necessarily identically distributed (non-iid) Rician channels [6] and the one-side correlated Rayleigh channels [7]. When there exist co-channel interferers in the system, the exact MGF of the isolated link capacity were obtained in [8][9]. The Gaussian approximated outage capacity were also obtained for above scenarios based on these MGF results. It was shown by simulations that this Gaussian approximation technique is very accurate even for a small number of antenna elements, for various fading scenarios, and with a wide range of transmitting signal-to-noise ratio (TSNR).

Since the exact outage capacity expressions for above cases are difficult to obtain explicitly, an accurate and simple approximation technique is highly desirable. The accuracy of the Gaussian approximation can be observed in general but a detailed study has not been done. In this paper, we quantify the approximation accuracy for various system configurations. Moreover, we also propose a Gamma approximation to the MIMO outage capacity to capture the positivity of the channel capacity and we compare the accuracy of this approximation to the Gaussian approximation.

The remainder of this paper is organized as follows. The next section we present the system model. The problem statement as well as some exact capacity results for MIMO channels in the presence of co-channel interferers are given in section III. In that section, we also review the Gaussian approximation technique proposed in [3] and propose the new Gamma approximation to the MIMO channel capacity CCDF. Finally, we present some numerical examples in section IV to study the accuracy of these two approximation methods for various system configurations.

II. System Model

We consider a single user Gaussian channel with T antenna elements at the transmitter and R antenna elements at the receiver. In general, a practical cellular system usually has co-channel interference from the neighboring cells due to the frequency-reuse. The discrete equivalent $R \times 1$ received vector at the receiver can be modeled into the matrix

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form

$$\boldsymbol{y} = \boldsymbol{H}_D \boldsymbol{x}_D + \boldsymbol{H}_I \boldsymbol{P}_I^{\frac{1}{2}} \boldsymbol{x}_I + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{H}_D$ is the $R \times T$ channel matrix for the desired user and $\boldsymbol{x}_D$ is the transmitted vector with the power constraint

$$E\left(\boldsymbol{x}_D^H \boldsymbol{x}_D\right) \leq \Omega, \tag{2}$$

where $(\cdot)^H$ denotes the conjugate transpose operator and $E(\cdot)$ is the expected value. We assume that there are N_I co-channel interferers in the system and therefore in (1), $\boldsymbol{H}_I$ and $\boldsymbol{x}_I$ are the $R \times N_I$ channel matrix and the $N_I \times 1$ transmitted vector for the co-channel interferers. In (1), $\boldsymbol{P}_I = \text{diag}(P_1, P_2, \cdots, P_{N_I})$ denotes the average power matrix for the co-channel interferers. Without loss of generality, we assume that $P_1 > P_2 > \cdots > P_{N_I}$. A more detailed description and justification of this model can be found in [8]. In (1), $\boldsymbol{n}$ is the complex additive white Gaussian noise (AWGN) vector with zero mean and covariance matrix $\sigma_n^2 \boldsymbol{I}_R$, where $\boldsymbol{I}_R$ is the $R \times R$ identity matrix.

III. MIMO Capacity

When the CSI for both the desired user and co-channel interferers are known at the desired user's receiver but the transmitter has no CSI, the instantaneous capacity of the desired link for such channel is given by [10]

$$\begin{aligned} C &= \log_2\left(\det\left(\boldsymbol{I}_R + \frac{\Omega_D}{T}\boldsymbol{H}_D\boldsymbol{H}_D^H\boldsymbol{B}_I^{-1}\right)\right) \\ &= \sum_{k=1}^{\min(T,R)} \log_2\left(1 + \frac{\Omega_D}{T}\phi_k\right), \end{aligned} \tag{3}$$

where $0 < \phi_1 < \phi_2 < \cdots < \phi_{\min(T,R)}$ are the non-zero eigenvalues of $\left(\boldsymbol{B}_I^{-\frac{1}{2}}\boldsymbol{H}_D\right)\left(\boldsymbol{B}_I^{-\frac{1}{2}}\boldsymbol{H}_D\right)^H$. Note that $\left(\boldsymbol{B}_I^{-\frac{1}{2}}\boldsymbol{H}_D\right)\left(\boldsymbol{B}_I^{-\frac{1}{2}}\boldsymbol{H}_D\right)^H$ has the same non-zero eigenvalues as those of $\boldsymbol{F} = \boldsymbol{H}_D^H\boldsymbol{B}_I^{-1}\boldsymbol{H}_D = \boldsymbol{H}_D^H\left(\boldsymbol{H}_I\boldsymbol{P}_I\boldsymbol{H}_I^H + \sigma_n^2\boldsymbol{I}_R\right)^{-1}\boldsymbol{H}_D$. For fading channels, $\boldsymbol{H}_D$ and $\boldsymbol{H}_I$ in (3) are random matrices and therefore the capacity C is itself a random variable.

The exact capacity MGF when both the desired user and co-channel interferers are subject to Rayleigh fading was obtained in [8]. In such case, $\boldsymbol{H}_D : R \times T$ and $\boldsymbol{H}_I : R \times N_I$ are assumed to be independent complex Gaussian matrices whose entries are i.i.d. complex Gaussian random variables with zero mean and variance 1. We now use this MGF result to obtain the exact capacity mean and variance. Due to the space limitation, only the final desired results are provided here and the detailed derivations are omitted.

A. Mean Capacity

1) When $R \leq T$ and $N_I > R$:

$$\begin{aligned} E(C) &= \frac{\sum_{k=1}^{R}\det(\boldsymbol{G_1}(k))}{\ln(2)\prod_{k=1}^{R}\Gamma(T-k+1)\Gamma(R-k+1)}, \\ &\times \frac{1}{\det\left(P_i^{j-1}\right)} \end{aligned} \tag{4}$$

where $\Gamma(\cdot)$ is the gamma function, $\det\left(P_i^{j-1}\right)$ is the Vandermonde determinant defined by

$$\det\left(P_i^{j-1}\right) = \prod_{1 \leq l < k \leq N_I} (P_k - P_l), \tag{5}$$

and the $N_I \times N_I$ matrices $\boldsymbol{G_1}(k)$, $k = 1, \cdots, R$, are defined by

$$\boldsymbol{G_1}(k) = \begin{pmatrix} \boldsymbol{C_{1A}} \\ \boldsymbol{G_{1B}}(k) \end{pmatrix}, \tag{6}$$

where the $(N_I - R) \times N_I$ block $\boldsymbol{C_{1A}}$ is defined by

$$\{\boldsymbol{C_{1A}}\}_{i,j} = P_j^{i-1}, i = 1, \cdots, N_I - R;\ j = 1, \cdots, N_I \tag{7}$$

and the $R \times N_I$ block $\boldsymbol{G_{1B}}(k)$ is defined by

$$\begin{aligned} \{\boldsymbol{G_{1B}}(k)\}_{i=k,j} &= P_j^{N_I-R-1}e^{\sigma_n^2/P_j}\int_0^\infty \frac{\ln(1+\rho y)y^{T-i}}{(1/P_j+y)^{T+1}} \\ &\times \Gamma\left(T+1, \sigma_n^2\left(1/P_j+y\right)\right)dy, \\ &\quad i = 1, \cdots, R;\ j = 1, \cdots, N_I \\ \{\boldsymbol{G_{1B}}(k)\}_{i\neq k,j} &= P_j^{N_I-R+i-1}\Gamma(T-i+1) \\ &\times e^{\frac{\sigma_n^2}{P_j}}\Gamma(i, \sigma_n^2/P_j), \\ &\quad j = 1, \cdots, N_I, \end{aligned} \tag{8}$$

where $\Gamma(\cdot,\cdot)$ is the complementary incomplete gamma function defined by [11, Eqn. (8.350.2)].

2) When $R \leq T$ and $N_I \leq R$:

$$\begin{aligned} E(C) = &\frac{\left(\sigma_n^2\right)^{T(R-N_I)}}{\ln(2)\left(\prod_{k=1}^{R}\Gamma(T-k+1)\Gamma(R-k+1)\right)} \\ &\times \frac{\sum_{k=1}^{R}\det(\boldsymbol{G_2}(k))}{\det\left(P_i^{j-1}\right)\left(\prod_{n=1}^{N_I}P_n^{R-N_I+1}\right)}, \end{aligned} \tag{9}$$

where the $R \times R$ matrices $\boldsymbol{G_2}(k)$, $k = 1, \cdots, R$, are defined by

$$\boldsymbol{G_2}(k) = \begin{pmatrix} \boldsymbol{G_{2A}}(k) \\ \boldsymbol{G_{2B}}(k) \end{pmatrix}, \tag{10}$$

where the $(R - N_I) \times R$ block $\boldsymbol{G_{2A}}(k)$ is given by

$$\begin{aligned} \{\boldsymbol{G_{2A}}(k)\}_{i,j=k} &= (-1)^{i-1}\Gamma(T+i-j)e^{\sigma_n^2/\rho} \\ &\times \sum_{m=1}^{T+i-j}\frac{\Gamma(m-(T+i-j), \sigma_n^2/\rho)}{\sigma_n^{2m}\rho^{T+i-j-m}}, \\ \{\boldsymbol{G_{2A}}(k)\}_{i,j\neq k} &= (-1)^{i-1}\Gamma(T+i-j)\frac{1}{\sigma_n^{2(T+i-j)}} \\ &\quad i = 1, \cdots, R - N_I \end{aligned} \tag{11}$$

and the $N_I \times R$ block $\boldsymbol{G_{2B}}(k)$ is defined by

$$\begin{aligned} &\{\boldsymbol{G_{2B}}(k)\}_{i,j=k} = \Gamma(T+1)\sum_{m=0}^{T}\frac{(\sigma_n^2)^m}{m!} \\ &\times \int_0^\infty \ln(1+\rho y)\frac{y^{T-j}e^{-y\sigma_n^2}}{(1/P_i+y)^{(T+1-m)}}dy, \\ &\{\boldsymbol{G_{2B}}(k)\}_{i,j\neq k} = \Gamma(T-j+1)e^{\frac{\sigma_n^2}{P_i}}P_i^j\Gamma(j, \sigma_n^2/P_i) \\ &i = 1, \cdots, N_I. \end{aligned} \tag{12}$$

3) When $R > T$ and $N_I \geq R$:

$$E(C) = \frac{(-1)^{T(R-T)}}{\ln(2)\left(\prod_{k=1}^{R}\Gamma(R-k+1)\right)} \times \frac{\sum_{k=1}^{T}\det(\boldsymbol{G_3}(k))}{\left(\prod_{k=1}^{T}\Gamma(T-k+1)\right)\det\left(P_i^{j-1}\right)}, \quad (13)$$

where the $N_I \times N_I$ matrices $\boldsymbol{G_3}(k)$, $k = 1, \cdots, T$, are given by

$$\boldsymbol{G_3}(k) = \begin{pmatrix} \boldsymbol{C_{3A}} \\ \boldsymbol{C_{3B}} \\ \boldsymbol{G_{3C}}(k) \end{pmatrix}, \quad (14)$$

where the $(N_I - R) \times N_I$ block $\boldsymbol{C_{3A}}$ is defined by

$$\{\boldsymbol{C_{3A}}\}_{i,j} = P_j^{i-1}, \quad i = 1, \cdots, N_I - R;\ j = 1, \cdots, N_I, \quad (15)$$

the $(R-T) \times N_I$ block $\boldsymbol{C_{3B}}$ is defined by

$$\{\boldsymbol{C_{3B}}\}_{i,j} = P_j^{N_I-R+T+i-1} e^{\frac{\sigma_n^2}{P_j}} \Gamma\left(T+i, \frac{\sigma_n^2}{P_j}\right), \quad i = 1, \cdots, R-T;\ \ j = 1, \cdots, N_I, \quad (16)$$

and the $T \times N_I$ block $\boldsymbol{G_{3C}}(k)$ is given by

$$\{\boldsymbol{G_{3C}}(k)\}_{i=k,j} = P_j^{N_I-R-1}\Gamma(T+1)\sum_{m=0}^{T}\frac{(\sigma_n^2)^m}{m!} \times \int_0^\infty \ln(1+\rho y)\frac{y^{T-i}e^{-y\sigma_n^2}}{(1/P_j+y)^{(T+1-m)}}dy,$$

$$\{\boldsymbol{G_{3C}}(k)\}_{i\neq k,j} = P_j^{N_I-R-1}\Gamma(T-i+1)e^{\frac{\sigma_n^2}{P_j}}P_j^i\Gamma(i,\sigma_n^2/P_j) \quad i = 1, \cdots, T;\ \ j = 1, \cdots, N_I, \quad (17)$$

4) When $R \geq N_I > T$:

$$E(C) = \frac{(-1)^{T(R-T)}(\sigma_n^2)^{T(R-N_I)}}{\ln(2)\left(\prod_{k=1}^{T}\Gamma(T-k+1)\right)} \times \frac{\sum_{k=1}^{T}\det(\boldsymbol{G_4}(k))}{\left(\prod_{k=1}^{N_I}\Gamma(R-k+1)P_k^{R-N_I+1}\right)\det\left(P_i^{j-1}\right)} \quad (18)$$

where the $N_I \times N_I$ matrices $\boldsymbol{G_4}(k)$, $k = 1, \cdots, T$, are given by

$$\boldsymbol{G_4}(k) = \begin{pmatrix} \boldsymbol{C_{4A}} \\ \boldsymbol{G_{4B}}(k) \end{pmatrix}, \quad (19)$$

where the $(N_I - T) \times N_I$ block $\boldsymbol{C_{4A}}$ is defined by

$$\{\boldsymbol{C_{4A}}\}_{i,j} = \sum_{k=0}^{T}\binom{T}{k}(\sigma_n^2)^{T-k}P_j^{R-N_I+i+k} \times \Gamma(R-N_I+i+k), \quad i = 1, \cdots, N_I - T;\ j = 1, \cdots, N_I, \quad (20)$$

and the $T \times N_I$ block $\boldsymbol{G_{4B}}(k)$ is defined by

$$\{\boldsymbol{G_{4B}}(k)\}_{i=k,j} = \Gamma(T+1)\sum_{m=0}^{T}\frac{(\sigma_n^2)^m}{m!} \times \int_0^\infty \ln(1+\rho y)y^{T-i}e^{-y\sigma_n^2}\left(1/P_j+y\right)^{-(T+1-m)}dy - \sum_{l=0}^{R-N_I-1}\frac{(-1)^l}{l!}\Gamma(T-i+2)e^{\frac{\sigma_n^2}{\rho}} \times \left(\sum_{m=0}^{T}\binom{T}{m}(\sigma_n^2)^{T-m}P_j^{l+m+1}\Gamma(l+m+1)\right) \times \sum_{n=1}^{T-i+2}\frac{\Gamma(n-(T-i+2),\sigma_n^2/\rho)}{\sigma_n^{2n}\rho^{T-i+2-n}}$$

$$\{\boldsymbol{G_{4B}}(k)\}_{i\neq k,j} = \Gamma(T-i+1)e^{\frac{\sigma_n^2}{P_j}}P_j^i\Gamma(i,\sigma_n^2/P_j) - \sum_{l=0}^{R-N_I-1}\frac{(-1)^l}{l!}\Gamma(T-i+l+1)(\sigma_n^2)^{-(T-i+l+1)} \times \left(\sum_{m=0}^{T}\binom{T}{m}(\sigma_n^2)^{T-m}P_j^{l+m+1}\Gamma(l+m+1)\right) \quad j = 1, \cdots, N_I. \quad (21)$$

5) When $R \geq T$ and $N_I < T$:

$$E(C) = \frac{(-1)^{T(R-T)}(\sigma_n^2)^{T(R-N_I)}}{\ln(2)\left(\prod_{k=1}^{T}\Gamma(T-k+1)\Gamma(R-k+1)\right)} \times \frac{\sum_{k=1}^{T}\det(\boldsymbol{G_5}(k))}{\left(\prod_{k=1}^{N_I}P_k^{R-N_I+1}\right)\det\left(P_i^{j-1}\right)} \quad (22)$$

where the $T \times T$ matrices $\boldsymbol{G_5}(k)$, $k = 1, \cdots, T$, are given by

$$\boldsymbol{G_5}(k) = \begin{pmatrix} \boldsymbol{G_{5A}}(k) & \boldsymbol{G_{4B}}(k) \end{pmatrix}, \quad (23)$$

where the $T \times (T - N_I)$ block $\boldsymbol{G_{5A}(k)}$ is defined by

$$\{\boldsymbol{G_{5A}}(k)\}_{i=k,j} = (-1)^{R-T+j-1}\Gamma(R-i+j)e^{\sigma_n^2/\rho} \times \sum_{m=1}^{R-i+j}\frac{\Gamma(m-(R-i+j),\sigma_n^2/\rho)}{\sigma_n^{2m}\rho^{(R-i+j-m)}}$$

$$\{\boldsymbol{G_{5A}}(k)\}_{i\neq k,j} = (-1)^{R-T+j-1}\Gamma(R-i+j) \times (\sigma_n^2)^{-(R-i+j)} \quad j = 1, \cdots, T - N_I \quad (24)$$

and the $T \times N_I$ block $\boldsymbol{G_{4B}}(k)$ is defined by (21).

B. Capacity Variance

Since

$$\mathrm{Var}(C) = E\left(C^2\right) - (E(C))^2, \quad (25)$$

we now only need to obtain the second moment $E\left(C^2\right)$ to compute the capacity variance. We first describe our notations: We will use $\{\boldsymbol{D}(k,l)\}_{k=l,j}$ to denote the entries at the ith row and jth column when $i = k = l$ (i.e. the second derivative is taken with respect to τ), $\{\boldsymbol{D}(k,l)\}_{k\neq l,j}$ to denote the entries at the ith row and jth column when $k \neq l$ and $i = k$ or $i = l$ (first derivative is taken with respect to τ), and $\{\boldsymbol{D}(k,l)\}_{i,j}$ to denote the entries at the ith row and jth column when $i \neq k$ and $i \neq l$ (no derivative is taken). In some cases, the derivatives are taken column by column, in which cases, we will use the similar notations except that j will be used as the indicator.

1) When $R \leq T$ and $N_I > R$:

$$E\left(C^2\right) = \frac{\sum_{k,l=1}^{R} \det(\boldsymbol{G_1}(k,l))}{\ln^2(2)\prod_{k=1}^{R}\Gamma(T-k+1)\Gamma(R-k+1)} \times \frac{1}{\det\left(P_i^{j-1}\right)} \tag{26}$$

where the $N_I \times N_I$ matrices $\boldsymbol{G_1}(k,l)$, $k,l = 1,\cdots,R$, are defined by

$$\boldsymbol{G_1}(k,l) = \begin{pmatrix} \boldsymbol{C_{1A}} \\ \boldsymbol{G_{1B}}(k,l) \end{pmatrix}, \tag{27}$$

where the $(N_I - R) \times N_I$ block $\boldsymbol{C_{1A}}$ is defined by (7) and the $R \times N_I$ block $\boldsymbol{G_{1B}}(k,l)$ is defined by

$$\begin{aligned} \{\boldsymbol{G_{1B}}(k,l)\}_{k=l,j} &= P_j^{N_I-R-1} e^{\sigma_n^2/P_j} \int_0^\infty \frac{\ln^2(1+\rho y) y^{T-i}}{(1/P_j+y)^{T+1}} \\ &\times \ \Gamma\left(T+1, \sigma_n^2\left(1/P_j+y\right)\right) dy, \\ &\quad i = 1,\cdots,R; \ j = 1,\cdots,N_I \\ \{\boldsymbol{G_{1B}}(k,l)\}_{k\neq l,j} &= \{\boldsymbol{G_{1B}}(k)\}_{i=k,j} \\ \{\boldsymbol{G_{1B}}(k,l)\}_{i,j} &= \{\boldsymbol{G_{1B}}(k,l)\}_{i\neq k,j} \\ &\quad j = 1,\cdots,N_I, \end{aligned} \tag{28}$$

where $\{\boldsymbol{G_{1B}}(k)\}_{i=k,j}$ and $\{\boldsymbol{G_{1B}}(k)\}_{i\neq k,j}$ are given by (8).

2) When $R \leq T$ and $N_I \leq R$:

$$\begin{aligned} E\left(C^2\right) &= \frac{\left(\sigma_n^2\right)^{T(R-N_I)}}{\ln^2(2)\left(\prod_{k=1}^{R}\Gamma(T-k+1)\Gamma(R-k+1)\right)} \\ &\times \frac{\sum_{k,l=1}^{R}\det(\boldsymbol{G_2}(k,l))}{\left(\prod_{n=1}^{N_I} P_n^{R-N_I+1}\right)\det\left(P_i^{j-1}\right)} \end{aligned} \tag{29}$$

where the $R \times R$ matrices $\boldsymbol{G_2}(k,l)$, $k,l = 1,\cdots,R$, are defined by

$$\boldsymbol{G_2}(k,l) = \begin{pmatrix} \boldsymbol{G_{2A}}(k,l) \\ \boldsymbol{G_{2B}}(k,l) \end{pmatrix}, \tag{30}$$

where the $(R - N_I) \times R$ block $\boldsymbol{G_{2A}}(k,l)$ is given by

$$\begin{aligned} \{\boldsymbol{G_{2A}}(k,l)\}_{i,k=l} &= (-1)^{i-1} \\ &\times \int_0^\infty \ln^2(1+\rho y) y^{T+i-k-1} e^{-y\sigma_n^2} dy \\ \{\boldsymbol{G_{2A}}(k,l)\}_{i,k\neq l} &= \{\boldsymbol{G_{2A}}(k)\}_{i,j=k} \\ \{\boldsymbol{G_{2A}}(k,l)\}_{i,j} &= \{\boldsymbol{G_{2A}}(k)\}_{i,j\neq k} \\ &\quad i = 1,\cdots,R-N_I, \end{aligned} \tag{31}$$

where $\{\boldsymbol{G_{2A}}(k)\}_{i,j\neq k}$ and $\{\boldsymbol{G_{2A}}(k)\}_{i,j\neq k}$ are given by (11), and the $N_I \times R$ block $\boldsymbol{G_{2B}}(k,l)$ is defined by

$$\begin{aligned} &\{\boldsymbol{G_{2B}}(k,l)\}_{i,k=l} = \Gamma(T+1)\sum_{m=0}^{T}\frac{(\sigma_n^2)^m}{m!} \\ &\times \int_0^\infty \ln^2(1+\rho y) y^{T-j} e^{-y\sigma_n^2}\left(1/P_i+y\right)^{-(T+1-m)} dy \\ &\{\boldsymbol{G_{2B}}(k,l)\}_{i,k\neq l} = \{\boldsymbol{G_{2B}}(k)\}_{i,j=k} \\ &\{\boldsymbol{G_{2B}}(k,l)\}_{i,j} = \{\boldsymbol{G_{2B}}(k)\}_{i,j\neq k} \\ &i = 1,\cdots,N_I, \end{aligned} \tag{32}$$

where $\{\boldsymbol{G_{2B}}(k)\}_{i,j\neq k}$ and $\{\boldsymbol{G_{2B}}(k)\}_{i,j\neq k}$ are given by (12).

3) When $R > T$ and $N_I \geq R$:

$$\begin{aligned} E\left(C^2\right) &= \frac{(-1)^{T(R-T)}}{\ln^2(2)\left(\prod_{k=1}^{R}\Gamma(R-k+1)\right)} \\ &\times \frac{\sum_{k,l=1}^{T}\det(\boldsymbol{G_3}(k,l))}{\left(\prod_{k=1}^{T}\Gamma(T-k+1)\right)\det\left(P_i^{j-1}\right)} \end{aligned} \tag{33}$$

where $\boldsymbol{G_3}(k,l)$, $k,l = 1,\cdots,T$, are $N_I \times N_I$ matrices given by

$$\boldsymbol{G_3}(k,l) = \begin{pmatrix} \boldsymbol{C_{3A}} \\ \boldsymbol{C_{3B}} \\ \boldsymbol{G_{3C}}(k,l) \end{pmatrix}, \tag{34}$$

where the $(N_I - R) \times N_I$ block $\boldsymbol{C_{3A}}$ is defined by (15), the $(R - T) \times N_I$ block $\boldsymbol{C_{3B}}$ is defined by (16), and the $T \times N_I$ block $\boldsymbol{G_{3C}}(k,l)$ is given by

$$\begin{aligned} &\{\boldsymbol{G_{3C}}(k,l)\}_{k=l,j} = P_j^{N_I-R-1}\Gamma(T+1)\sum_{m=0}^{T}\frac{(\sigma_n^2)^m}{m!} \\ &\times \int_0^\infty \ln^2(1+\rho y) y^{T-i} e^{-y\sigma_n^2}\left(1/P_j+y\right)^{-(T+1-m)} dy \\ &\{\boldsymbol{G_{3C}}(k,l)\}_{k\neq l,j} = \{\boldsymbol{G_{3C}}(k)\}_{i=k,j} \\ &\{\boldsymbol{G_{3C}}(k,l)\}_{i,j} = \{\boldsymbol{G_{3C}}(k)\}_{i\neq k,j} \\ &j = 1,\cdots,N_I, \end{aligned} \tag{35}$$

where $\{\boldsymbol{G_{3C}}(k)\}_{i=k,j}$ and $\{\boldsymbol{G_{3C}}(k)\}_{i\neq k,j}$ are given by (17).

4) When $R \geq N_I > T$:

$$\begin{aligned} E\left(C^2\right) &= \frac{(-1)^{T(R-T)}(\sigma_n^2)^{T(R-N_I)}}{\ln^2(2)\left(\prod_{k=1}^{T}\Gamma(T-k+1)\right)} \\ &\times \frac{\sum_{k,l=1}^{T}\det(\boldsymbol{G_4}(k,l))}{\left(\prod_{k=1}^{N_I}\Gamma(R-k+1)P_k^{R-N_I+1}\right)\det\left(P_i^{j-1}\right)}, \end{aligned} \tag{36}$$

where $\boldsymbol{G_4}(k,l)$ is an $N_I \times N_I$ matrix given by

$$\boldsymbol{G_4}(k,l) = \begin{pmatrix} \boldsymbol{C_{4A}} \\ \boldsymbol{G_{4B}}(k,l) \end{pmatrix}, \tag{37}$$

where the $(N_I - T) \times N_I$ block $\boldsymbol{C_{4A}}$ is defined by (20) and the $T \times N_I$ block $\boldsymbol{G_{4B}}(k,l)$ is defined by

$$\begin{aligned}
&\{\boldsymbol{G_{4B}}(k,l)\}_{k=l,j} = \Gamma(T+1) \sum_{m=0}^{T} \frac{(\sigma_n^2)^m}{m!} \\
&\times \int_0^\infty \ln^2(1+\rho y) y^{T-i} e^{-y\sigma_n^2} \left(1/P_j + y\right)^{-(T+1-m)} dy \\
&- \sum_{l=0}^{R-N_I-1} \frac{(-1)^l}{l!} \int_0^\infty \ln^2(1+\rho y) y^{T-i+l} e^{-\sigma_n^2 y} dy \\
&\times \left(\sum_{m=0}^{T} \binom{T}{m} (\sigma_n^2)^{T-m} P_j^{l+m+1} \Gamma(l+m+1) \right) \\
&\{\boldsymbol{G_{4B}}(k,l)\}_{k\neq l,j} = \{\boldsymbol{G_{4B}}(k)\}_{i=k,j} \\
&\{\boldsymbol{G_{4B}}(k,l)\}_{i,j} = \{\boldsymbol{G_{4B}}(k)\}_{i\neq k,j} \\
&j = 1, \cdots, N_I,
\end{aligned} \tag{38}$$

where $\{\boldsymbol{G_{4B}}(k)\}_{i=k,j}$ and $\{\boldsymbol{G_{4B}}(k)\}_{i\neq k,j}$ are given by (21).

5) When $R \geq T$ and $N_I < T$:

$$\begin{aligned}
E\left(C^2\right) &= \frac{(-1)^{T(R-T)} (\sigma_n^2)^{T(R-N_I)}}{\ln^2(2) \left(\prod_{k=1}^{T} \Gamma(T-k+1)\Gamma(R-k+1) \right)} \\
&\times \frac{\sum_{k,l=1}^{T} \det(\boldsymbol{G_5}(k,l))}{\left(\prod_{k=1}^{N_I} P_k^{R-N_I+1} \right) \det\left(P_i^{j-1} \right)},
\end{aligned} \tag{39}$$

where $\boldsymbol{G_5}(k,l)$, $k,l = 1, \cdots, T$, are $T \times T$ matrices given by

$$\boldsymbol{G_5}(k,l) = \left(\begin{array}{cc} \boldsymbol{G_{5A}}(k,l) & \boldsymbol{G_{4B}}(k,l) \end{array} \right), \tag{40}$$

where the $T \times (T - N_I)$ block $\boldsymbol{G_{5A}}(k,l)$ is defined by

$$\begin{aligned}
&\{\boldsymbol{G_{5A}}(k,l)\}_{k=l,j} = (-1)^{R-T+j-1} \\
&\times \int_0^\infty \ln^2(1+\rho y) y^{R-i+j-1} e^{-\sigma_n^2 y} dy, \\
&\{\boldsymbol{G_{5A}}(k,l)\}_{k\neq l,j} = \{\boldsymbol{G_{5A}}(k)\}_{i=k,j} \\
&\{\boldsymbol{G_{5A}}(k,l)\}_{i,j} = \{\boldsymbol{G_{5A}}(k)\}_{i\neq k,j} \\
&j = 1, \cdots, T - N_I,
\end{aligned} \tag{41}$$

where $\{\boldsymbol{G_{5A}}(k)\}_{i=k,j}$ and $\{\boldsymbol{G_{5A}}(k)\}_{i\neq k,j}$ are defined by (24), and the $T \times N_I$ block $\boldsymbol{G_{4B}}(k,l)$ is defined by (38).

C. Gaussian and Gamma Approximations

The capacity CCDF is the probability the capacity exceeds a particular capacity threshold C_{th}. Smith and Shafi observed in [3] that the MIMO channel capacity is approximately Gaussian distributed. They then derived the capacity variance for the i.i.d. Rayleigh fading scenario to obtain a Gaussian approximated capacity CCDF [3]. This Gaussianity of the MIMO channel capacity was also independently observed in [4]. Later, the MGF of the capacity without CSI at the transmitter has been derived for the i.i.d. Rayleigh [4][5][6], non-i.i.d. Rician [6], and the one-side correlated Rayleigh (either rows or columns of $\boldsymbol{H}$ are correlated but not both) [7] fading scenarios. In [12], the capacity MGF was also obtained for the MIMO Rayleigh channels with covariance feedback. In all above cases, the mean and variance of capacity were derived based on the MGF results and were used to obtain the outage capacity with the help of the Gaussian approximation to the MIMO channel capacity proposed in [3][4].

With the capacity mean and variance in hand, we can get the Gaussian approximated capacity CCDF. It is also possible to approximate the MIMO capacity by a Gamma random variable, which can also be determined by its first two moments. Note that the positivity of the channel capacity is captured by a Gamma random variable. For a Gamma random variable x, Gamma(α,θ), its PDF is given by

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\theta)\theta^{\alpha}} e^{-\frac{x}{\theta}}, \tag{42}$$

while its mean is given by $\alpha\theta$ and its variance is given by $\alpha\theta^2$. Therefore, if we approximate the capacity by a Gamma RV, the parameters of this Gamma RV can be easily determined as

$$\theta = \frac{\text{Var}(C)}{E(C)} \tag{43}$$

and

$$\alpha = \frac{(E(C))^2}{\text{Var}(C)}. \tag{44}$$

The Gamma approximated CCDF of the capacity is simply given by

$$C_{\text{ccdf}} = 1 - \frac{\gamma(\alpha, C_{\text{th}}/\theta)}{\Gamma(\alpha)}, \tag{45}$$

where $\gamma(\cdot,\cdot)$ is incomplete gamma function defined by [11, Eqn. (8.350.1)]. In the next section, we will study and compare the accuracy of the Gaussian and the Gamma approximations.

IV. Some Numerical Examples

In Fig. 1, we use the results of [6] to investigate the accuracy of the Gaussian and Gamma approximations in a Rician fading environment. More specifically, Fig, 1 compares the capacity CCDF $\Pr\left(C \geq C_{\text{th}}\right)$ obtained in absence of co-channel interference with (i) the Gaussian approximation, (ii) the Gamma approximation, and (iii) simulations, for i.i.d. Rician MIMO channels with Rician factor $K = 4$ when TSNR= 0 dB with $T = R = 2, 4, 6,$ and 8. It can be seen that both Gamma and Gaussian approximations are very accurate even for small numbers of antenna elements. To get a more specific idea about the accuracy of these two approximation methods, we tabulate in Table 1 the capacity CCDF values obtained in Fig. 1. Due to the space limitation, we omit the case when $T = R = 8$. It shows that indeed the two approximations are very accurate even for a 2×2 antenna array and a close look at the data reveals that in general the Gamma approximation method yields slightly more accurate results than the Gaussian approximation for a Rician MIMO channel, especially when the TSNR is not high. Fig. 2 is similar to Fig. 1 except that the TSNR in Fig. 2 is 10 dB. Both Fig. 1 and 2 indicate that the Gamma and Gaussian approximation work very well for even small numbers of antenna elements and for a wide range of TSNR values.

In Fig. 3, we use the mean and variance results of the section III to obtain Gaussian and Gamma approximations to the CCDF of MIMO capacity in presence of interference. More specifically, Fig. 3 plots the capacity CCDF $\Pr(C \geq C_{\mathrm{th}})$ obtained with (i) the Gaussian approximation, (ii) the Gamma approximation, and (iii) simulations, for i.i.d. Rayleigh MIMO channels with $T = R = 3$, $N_I = 0, 1, 2, 3, 4$, $\sigma_n^2 = 1$, and TSNR= 10 dB. The average powers of Rayleigh interferers are chosen to be the N_I largest elements from the set $\{1.5, 1.6, 1.8, 2.0\}$. This figure again shows that both the Gaussian and Gamma approximations work very well even in the presence of co-channel interference.

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C_{th}	$T = R = 2$			$T = R = 4$			$T = R = 6$		
(bps/Hz)	Sim.	Gaus.	Gam.	Sim.	Gaus.	Gam.	Sim.	Gaus.	Gam.
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.5	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.0	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.5	0.99996	0.99997	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.0	0.99931	0.99924	0.99994	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.5	0.99145	0.98949	0.99528	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.0	0.93348	0.92550	0.93476	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.5	0.71772	0.71861	0.70739	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.0	0.36965	0.38756	0.37002	1.00000	0.99998	1.00000	1.00000	1.00000	1.00000
4.5	0.12495	0.12506	0.12731	0.99985	0.99941	0.99983	1.00000	1.00000	1.00000
5.0	0.02920	0.02198	0.02941	0.99359	0.98991	0.99327	1.00000	1.00000	1.00000
5.5	0.00432	0.00200	0.00477	0.92455	0.91976	0.92445	1.00000	1.00000	1.00000
6.0	0.00039	0.00009	0.00057	0.67655	0.68578	0.67789	1.00000	1.00000	1.00000
6.5				0.32305	0.33154	0.32307	0.99995	0.99980	0.99994
7.0				0.09130	0.08767	0.09108	0.99717	0.99563	0.99715
7.5				0.01509	0.01146	0.01505	0.95966	0.95591	0.95997
8.0				0.00148	0.00070	0.00150	0.78184	0.78473	0.78243
8.5							0.44000	0.44885	0.44047
9.0							0.14840	0.14792	0.14820
9.5							0.02829	0.02487	0.02831
10.0							0.00294	0.00199	0.00308
10.5							0.00015	0.00007	0.00020

Table 1. Comparison of the capacity CCDF obtained with (i) simulation (Sim.), (ii) the Gaussian approximation (Gaus.), and (iii) the Gamma approximation (Gam.), for i.i.d. Rician MIMO channels when the Rician factor $K = 4$ and TSNR= 0 dB.

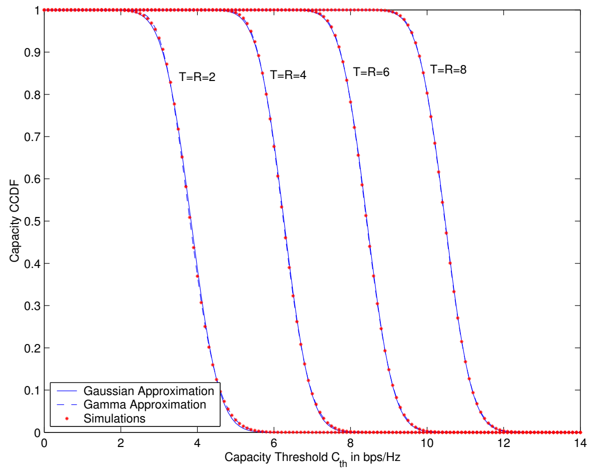

Fig. 1. Comparison of capacity CCDF $\Pr(C \geq C_{\mathrm{th}})$ obtained with (i) the Gaussian approximation, (ii) the Gamma approximation, and (iii) simulations, for i.i.d. Rician fading channels with Rician factor $K = 4$ when TSNR= 0 dB.

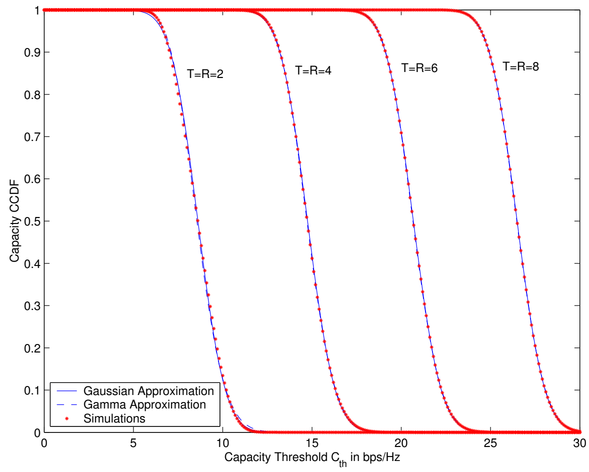

Fig. 2. Comparison of capacity CCDF $\Pr(C \geq C_{\mathrm{th}})$ obtained with (i) the Gaussian approximation, (ii) the Gamma approximation, and (iii) simulations, for i.i.d. Rician fading channels with Rician factor $K = 4$ when TSNR= 10 dB.

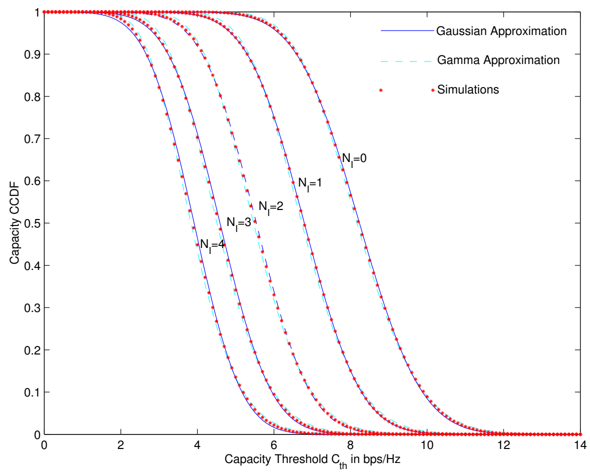

Fig. 3. Comparison of capacity CCDF $\Pr(C \geq C_{\mathrm{th}})$ obtained with (i) the Gaussian approximation, (ii) the Gamma approximation, and (iii) simulations, for i.i.d. Rayleigh MIMO channels with $T = R = 3$, $N_I = 0, 1, 2, 3, 4$, $\sigma_n^2 = 1$, and TSNR= 10 dB. The average powers of Rayleigh interferers are chosen to be the N_I largest elements from the set $\{1.5, 1.6, 1.8, 2.0\}$.