ON THE SHANNON CAPACITY OF DUAL MIMO SYSTEMS IN RICEAN FADING

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ABSTRACT

We derive exact analytical expressions for the probability density function and cumulative distribution function for the capacity of a dual multiple-input multiple-output system (either two transmit or two receive antennas) transmitting in Ricean fading. In contrast to earlier work we do not require the *line-of-sight* (LOS) channel matrix to be of rank one. As an example, we investigate the special case of uniform linear arrays, and derive expressions for the eigenvalues of a possibly full rank LOS channel matrix for this case. The results are verified by comparing the analytical expressions with Monte Carlo simulations.

1. INTRODUCTION

Multiple-input multiple-output (MIMO) technology is a promising tool for enabling spectrally efficient future wireless applications. A lot of research effort has been put into the MIMO field since the pioneering work of Foschini and Gans [1] and Telatar [2]. The capacity of MIMO systems has been widely investigated, but for the non-Rayleigh case analytical investigations have been mainly concerned with the ergodic capacity, and most of the work concentrate on bounds and asymptotic scenarios [3], [4], [5].

In [6] the exact capacity *probability density function* (PDF) and *cumulative distribution function* (CDF) of dual MIMO systems exposed to Ricean fading are derived, under the assumption of a *line-of-sight* (LOS) channel matrix of rank one. In this paper we generalize the work in [6] by deriving the capacity PDF and the CDF for a more general case, where the LOS matrix can have full rank. This is useful for example for high-frequency fixed wireless access systems [7]. The derivations are split into two cases, the case where the LOS eigenvalues are unequal, and the case where they are equal. For the latter case special care is needed because the eigenvalue distribution function used in [6] is not defined.

As an example we look at MIMO transmission applying *uniform linear arrays* (ULAs) at both the transmitter and receiver. The general geometrical model introduced by the authors in [8], is used to model the LOS matrix. The eigenvalues of this matrix are derived for the dual MIMO system under investigation.

The rest of the paper is organized as follows. Section 2 describes the system model used. In section 3 we extend the results from [6] and [8], and derive the capacity statistics and associated LOS channel matrix eigenvalues for a dual MIMO system with no constraints on the rank of the matrix. The analytical results are verified by simulations in section 4, while conclusions are drawn in section 5.

2. SYSTEM MODEL

A MIMO transmission system employs several transmit and receive antennas when transmitting information over a wireless channel. We denote the number of transmitters by N, while the number of receivers is denoted by M. It is common to model the MIMO transmission over a frequency flat channel in baseband as

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n}, \tag{1}$$

where $\mathbf{r}$ is the $M \times 1$ received signal vector, $\mathbf{s}$ is the $N \times 1$ transmitted symbol vector, $\mathbf{H}$ is the $M \times N$ channel matrix, and $\mathbf{n}$ is the $M \times 1$ additive white Gaussian noise vector. The additive noise vector contains i.i.d. circular symmetric complex Gaussian elements with zero mean and variance σ_n^2. We assume slowly varying fading, i.e. the elements of the channel matrix are complex values which can be regarded as constant during the transmission of one symbol vector.

It is common to model the channel matrix as a sum of two components, a LOS component and a *non-LOS* (NLOS) component. This model is referred to as a *Ricean* channel model. The ratio between the power of the two components gives the *Ricean* K factor. Thus, the channel matrix can be expressed as

$$\mathbf{H} = a\mathbf{H_{LOS}} + b\mathbf{H_{NLOS}}, \tag{2}$$

where $K = \frac{a^2}{b^2}$ [9]. We require the channel matrix to be normalized, i.e. the elements should have unit power, in order to make the average *signal-to-noise-ratio* (SNR) independent of the channel matrix. To achieve this, each element in $\mathbf{H_{LOS}}$ and $\mathbf{H_{NLOS}}$ has to have an average power of one, and $a^2 + b^2 = 1$. The NLOS matrix is thus modeled as a Rayleigh matrix with i.i.d. circular symmetric complex Gaussian elements with zero mean and variance 1, while the LOS matrix is discussed in section 3.2.

In this work we also assume that perfect channel knowledge is available in the receiver, while the channel is unknown at the transmitter. Applying equal power transmission, the Shannon capacity (in bits/s/Hz) for this system can then be expressed as

$$C = \log_2 \left[\prod_{i=1}^{U} \left(1 + \frac{\gamma b^2}{N} w_i \right) \right], \tag{3}$$

where $U = \min(M, N)$, γ is the average SNR for a received symbol on one receive antenna, and w_i are eigenvalues of the matrix $\mathbf{W}$ defined as

$$\mathbf{W} = \begin{cases} \frac{1}{b^2}\mathbf{H}\mathbf{H}^H & \text{for } M \geq N \\ \frac{1}{b^2}\mathbf{H}^H\mathbf{H} & \text{for } M > N \end{cases}, \tag{4}$$

$(\cdot)^H$ being the Hermitian transpose operator. To perform the derivations in section 3.1 we also need to define the associated LOS version of (4) [6],

$$\mathbf{M} = \begin{cases} \frac{a^2}{b^2}\mathbf{H_{LOS}}\mathbf{H_{LOS}}^H & \text{for } M \geq N \\ \frac{a^2}{b^2}\mathbf{H_{LOS}}^H\mathbf{H_{LOS}} & \text{for } M > N \end{cases}. \tag{5}$$

The eigenvalues of $\mathbf{M}$ are denoted $\{f_i\}_{i=1}^{U}$.

3. DERIVATIONS

To limit the analytical complexity we concentrate on dual MIMO systems in this paper, i.e. $U = 2$. This is particularly relevant in communication with small devices where the space for antennas is restricted (e.g. hand-held devices). In section 3.1 we derive the general analytical expressions for the capacity PDF and CDF for dual MIMO systems in Ricean fading. The expressions found are functions of the associated LOS eigenvalues (f_1 and f_2), which will be derived for the special case of ULAs in section 3.2.

3.1. Capacity PDF and CDF

This derivation follows a similar procedure to the one used in [6]. The difference is that we do not require the LOS channel matrix to be of rank one, which implies new calculations and new solutions. First of all, removing the requirement of rank one is equivalent to allowing f_2 to be nonzero, and thus we can not make the same simplifications as those in [6]. Second, special care has to be taken when $f_1 = f_2$ occurs, since the joint eigenvalue distribution utilized in [6] is not defined in this case. We will start by looking at the situation where $f_1 \neq f_2$ in section 3.1.1, before this special case is investigated in section 3.1.2.

3.1.1. Case 1: $f_1 \neq f_2$

The unordered joint eigenvalue distribution for a MIMO system where $U = 2$, and $f_1 \neq f_2$, is given in [6] as

$$\begin{aligned} f_{\mathbf{w}}(w_1, w_2) = {} & \frac{e^{-(f_1+f_2)}(w_1 w_2)^{V-2}(w_1 - w_2)e^{-(w_1+w_2)}}{2(f_1 - f_2)(f_1 w_1 f_2 w_2)^{\frac{V-2}{2}}} \\ & \cdot \Big[I_{V-2}(2\sqrt{f_1 w_1}) I_{V-2}(2\sqrt{f_2 w_2}) \\ & - I_{V-2}(2\sqrt{f_1 w_2}) I_{V-2}(2\sqrt{f_2 w_1}) \Big], \end{aligned} \tag{6}$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind [10, p. 262], and $V = \max(M, N)$.

For the dual case, the capacity expression in (3) becomes

$$C = \log_2 \left[\left(1 + \frac{w_1}{\alpha} \right) \left(1 + \frac{w_2}{\alpha} \right) \right], \tag{7}$$

where $\alpha = N/(\gamma b^2) = N(1 + K)/\gamma$. Equation (7) can further be rewritten as

$$w_1 = \left(\frac{2^c}{1 + \frac{w_2}{\alpha}} - 1 \right) \alpha. \tag{8}$$

We do a variable transformation by substituting (8) into (6) and multiplying with the *Jacobian*, $\frac{\partial w_1}{\partial c}$. Integrating this new joint PDF, $f_{c,w_2}(c, w_2)$, over w_2 results in the marginal PDF of the capacity:

$$\begin{aligned} f_C(c) = {} & \frac{\ln(2) 2^c \alpha^{V+1} e^{-(f_1+f_2)}}{2(f_1 - f_2)(f_1 f_2)^{\frac{V-2}{2}}} \int_1^{2^c} \left(\frac{2^c}{v^2} - 1 \right) \\ & \left(\left(\frac{2^c}{v} - 1 \right) (v - 1) \right)^{\frac{V-2}{2}} e^{-\left(\frac{2^c}{v} + v - 2\right)\alpha} \\ & \left[A_{2,2}(f_1, f_2, v) - A_{2,2}(f_2, f_1, v) \right] dv. \end{aligned} \tag{9}$$

Here $A_{i,j}(x, y, z)$ is defined as

$$\begin{aligned} A_{i,j}(x, y, z) \triangleq {} & I_{V-i} \left(2\sqrt{x \left(\frac{2^c}{z} - 1 \right) \alpha} \right) \\ & \cdot I_{V-j} \left(2\sqrt{y(z - 1)\alpha} \right). \end{aligned} \tag{10}$$

We have so far not been able to find a closed form solution of the integral in (9), and thus it is solved numerically.

To find the CDF of the capacity we need to integrate over $f_C(c)$ from zero to C. When we investigate this integral we recognize the *generalized Nuttall Q-function*, which is defined in [11, Eq. (86)] as

$$Q_{m,n}(y, z) = \int_z^{\infty} x^m \exp\left(-\frac{x^2 + y^2}{2} \right) I_n(yx) dx. \tag{11}$$

This is exploited when solving the integral and finding the CDF for our case, as

$$F_C(c) = \frac{\alpha^{\frac{V+2}{2}}}{2(f_1 - f_2)} \int_1^{2^C} (v-1)^{\frac{V-2}{2}} \frac{e^{-(v-1)\alpha}}{v} \left[\Lambda_2(f_2, v)\psi_{1,2}(f_1, v) - \Lambda_2(f_1, v)\psi_{1,2}(f_2, v)\right] dv, \tag{12}$$

with $\Lambda_i(x, y)$ and $\psi_i(x, y)$ being defined respectively as

$$\Lambda_i(x, y) \triangleq I_{V-i}\left(2\sqrt{x(y-1)\alpha}\right) e^{-x} x^{-\frac{V-2}{2}} \tag{13}$$

and

$$\psi_{i,j}(x, y) \triangleq y(2x)^{-\frac{V-2}{2}} \left[\frac{1}{2\alpha}\Phi_{V+i,V-j}(x, y) + (1-y)\Phi_{V-2+i,V-j}(x, y)\right], \tag{14}$$

with $\Phi_{i,j}(x, y)$ given by

$$\Phi_{i,j}(x, y) \triangleq Q_{i,j}\left(\sqrt{2x}, 0\right) - Q_{i,j}\left(\sqrt{2x}, \sqrt{2\alpha\left(\frac{2^c}{y} - 1\right)}\right). \tag{15}$$

3.1.2. *Case 2:* $f_1 = f_2$

The situation where the associated LOS eigenvalues are equal is very interesting as it results in the highest capacity for a given K factor, and will now be studied further. By inspection we observe that when the LOS eigenvalues are equal in (6), the $(f_1 - f_2)$ term in the denominator becomes zero, and thus the expression is not valid in this case. By further investigating (6), we see that the expressions including the modified Bessel functions in the brackets also becomes zero in this case. We define $f = f_2$ and $f_1 = f + \varepsilon$, and use *L'Hospital's rule* [10, p. 132] to solve for the case when $\varepsilon \to 0$. In the derivations the following relations are used [10, p. 265],

$$\frac{\partial}{\partial x} I_n(x) = \frac{1}{2}\left[I_{n-1}(x) + I_{n+1}(x)\right], \tag{16}$$

and

$$I_{n-1}(x) = \frac{2n}{x} I_n(x) + I_{n+1}(x). \tag{17}$$

The unordered joint eigenvalue distribution thus becomes

$$f_{\mathbf{w}}(w_1, w_2) = \frac{e^{-2f}(w_1 w_2)^{V-2}(w_1 - w_2)e^{-(w_1+w_2)}}{2f^{V-\frac{3}{2}}(w_1 w_2)^{\frac{V-2}{2}}} \cdot \left[\sqrt{w_1} I_{V-3}(2\sqrt{f w_1}) I_{V-2}(2\sqrt{f w_2}) - \sqrt{w_2} I_{V-3}(2\sqrt{f w_2}) I_{V-2}(2\sqrt{f w_1})\right], \tag{18}$$

which is clearly different from (6).

This joint distribution of w_1 and w_2 can now be used in an equivalent procedure to derive the capacity statistics for this special situation where $f_1 = f_2$. Thus the capacity PDF becomes

$$f_C(c) = \frac{\ln(2) 2^c \alpha^{V+\frac{3}{2}} e^{-2f}}{2f^{V-\frac{3}{2}}} \int_1^{2^c} \left(\frac{2^c}{v^2} - 1\right) \left(\left(\frac{2^c}{v} - 1\right)(v-1)\right)^{\frac{V-2}{2}} e^{-\left(\frac{2^c}{v} + v - 2\right)\alpha} \left[\sqrt{\left(\frac{2^c}{v} - 1\right)} \cdot A_{3,2}(f, f, v) - \sqrt{(v-1)} \cdot A_{2,3}(f, f, v)\right] dv, \tag{19}$$

and the capacity CDF becomes

$$F_C(c) = \frac{\alpha^{\frac{V+2}{2}}}{2\sqrt{f}} \int_1^{2^C} (v-1)^{\frac{V-2}{2}} \frac{e^{-(v-1)\alpha}}{v} \left[\frac{1}{\sqrt{2}} \cdot \Lambda_2(f, v)\psi_{2,3}(f, v) - \sqrt{(v-1)\alpha} \cdot \Lambda_3(f, v)\psi_{1,2}(f, v)\right] dv. \tag{20}$$

A special case of equal associated LOS eigenvalues occurs for a pure Rayleigh channel matrix. By setting $K\left(= \frac{a^2}{b^2}\right) = 0$ in (5), we see that the associated LOS eigenvalues become $f_1 = f_2 = 0$ (all the elements in $\mathbf{M}$ becomes zero). Equation (18) is not defined for $f = 0$, because it results in a division by zero, thus the results in this section is not valid for the Rayleigh case ($K = 0$). A discussion on the capacity statistics for Rayleigh channels can be found elsewhere in the literature, e.g. in [2].

3.1.3. *Some comments on the Nuttall Q-function*

In [12][1] a closed form expression for the Nuttall Q-function (Eq. (11)) is derived in terms of the Marcum Q-function and the modified Bessel function, for the case where the sum of its two indices is odd. By investigating (14) and (15) we find that for this requirement to be fulfilled in our case, the indices of $\psi_{i,j}$ have to have an odd sum. From (12) and (20) we see that this is true in both cases, and thus we can use the results from [12] to evaluate the expressions in closed form. We have not written out these expressions in the paper because they are quite involved and do not give additional insight; thus we refer the reader to [12] for closed form characterization.

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[1] There is a typo in [12, Eq. (1)]. The equation should be equal to (11).

3.2. LOS eigenvalues for the case of ULA

From the previous subsection we see that the PDF and the CDF are functions of f_1 and f_2 (or f). We therefore need to find expressions for these eigenvalues. When doing so we will restrict ourselves to the special case of ULAs. The results introduced by the authors in [8] are utilized, and further extended to general analytical expressions for the eigenvalues when $U = 2$. In [8] we showed that high MIMO capacity gain is achieved for a pure LOS channel when the antennas in the transmitter and receiver ULAs are distributed in space according to the following equation:

$$d_t d_r = \frac{R\lambda}{V \cos\theta_t \cos\theta_r}, \tag{21}$$

where d_t and d_r are the antenna separations at the transmitter and receiver respectively, R is the distance between the two arrays, λ is the carrier wavelength, and θ_t and θ_r are the spherical θ angle at the local coordinate system at the transmitter and receiver respectively. As the equation shows, the key design parameter is the product between d_t and d_r, referred to as the *antenna separation product* (ASP).

A *deviation factor* was introduced in [8] to illustrate how the performance is affected by choosing an ASP that deviates from the optimal value in (21). The deviation factor, denoted η, is defined as the ratio between the optimal ASP and the actual ASP, i.e. $\eta = \text{ASP}_{opt}/\text{ASP}$. For example a deviation factor of 2 ($\eta \approx 3$ dB) indicates that the ASP is half of the optimal value, and that d_t and/or d_r must be increased if the optimal performance is to be reached.

To evaluate the capacity expressions in section 3.1 for the ULA case we now need to find expressions for the eigenvalues of $\mathbf{M}$ based on the geometrical LOS model used in [8]. By using a ray tracing technique and proper normalization, the elements of $\mathbf{H_{LOS}}$ can be written as [8]

$$h_{m,n}^{LOS} = \exp\left(\frac{j2\pi}{\lambda} r_{m,n}\right), \tag{22}$$

where $r_{m,n}$ is the path length between transmit antenna n and receive antenna m. For $R \gg (N-1)d_t$ and $R \gg (M-1)d_r$, $r_{m,n}$ can be closely approximated as [8]

$$\begin{aligned} r_{m,n} \approx{} & R + m d_r \sin\theta_r \cos\phi_r - n d_t \sin\theta_t \\ & + \frac{(m d_r \sin\theta_r \sin\phi_r)^2}{2R} \\ & + \frac{(m d_r \cos\theta_r - n d_t \cos\theta_t)^2}{2R}, \end{aligned} \tag{23}$$

where ϕ_r is the second spherical angle at the local receiver coordinate system (the other being θ_r).

We now insert this channel matrix in (5) and solve for the eigenvalues of $\mathbf{M}$, i.e. $\det(\mathbf{M} - \mathbf{I_U} f) = 0$, or equivalently

$$\left(V - \frac{f}{K}\right)^2 - \sum_{i=0}^{V-1}\sum_{j=0}^{V-1} \exp\left[\frac{j2\pi}{V\eta}(j-i)\right] = 0. \tag{24}$$

Solving this equation with respect to f gives the following expressions for the eigenvalues:

$$f_1 = K\left(V + \frac{\sin(\pi V/\eta)}{\sin(\pi/\eta)}\right) \tag{25}$$

$$f_2 = K\left(V - \frac{\sin(\pi V/\eta)}{\sin(\pi/\eta)}\right). \tag{26}$$

Optimal ASP ($\eta = 0$ dB) gives $f_1 = f_2$. In this situation the capacity term derived in section 3.1.2 is used. As described in [8] the situation where $f_1 = f_2$ will also occur periodically for $\eta < 0$ dB, because of the periodic behavior of the optimal ASP solution (21). The values of η that give equal eigenvalues are dependent on V, and can be found, by investigating (25) and (26), to be

$$\eta = -10\log(q) \quad \text{dB}, \tag{27}$$

where q is the set of all positive integers that do not have V as a factor. Thus all of these deviation factors give maximum capacity, but the situation where $\eta = 0$ dB is the one that requires the smallest array size and is therefore considered the most interesting from an implementation perspective.

4. SIMULATION AND RESULTS

In this section we compare the analytical capacity CDF expressions (12) and (20) derived in section 3 with Monte Carlo simulations. Each of the simulated distribution curves is based on 50000 channel realizations.

In Fig. 1 the impact of the Ricean K factor and the deviation factor is illustrated. Both simulated (continuous curve) and analytical (circles) results are plotted. In accordance with [8] and [9], the figure shows the capacity gain introduced by using the optimal ASP (giving a full rank $\mathbf{H_{LOS}}$). The deviation factor is 0 dB for the optimal design case, while $\eta = 30$ dB approximates $\text{rank}(\mathbf{M}) = 1$, and is what we would get if the results from [6] were used. For a Rayleigh channel, corresponding to $K = -\infty$ dB, the capacity is independent of the LOS matrix, and as the K factor increases the capacity becomes more and more dependent on the LOS matrix and thus the deviation factor. The figure shows that the analytical results agree perfectly with the simulations.

Fig. 2 shows how the capacity distribution is changing when the number of transmit and receive antennas are changed. Since $U = 2$ and $V \geq 2$, an increase in V gives a diversity and/or array gain, which results in a logarithmic (not linear) increase in capacity. As the figure shows the analytical results follow the simulated results perfectly.

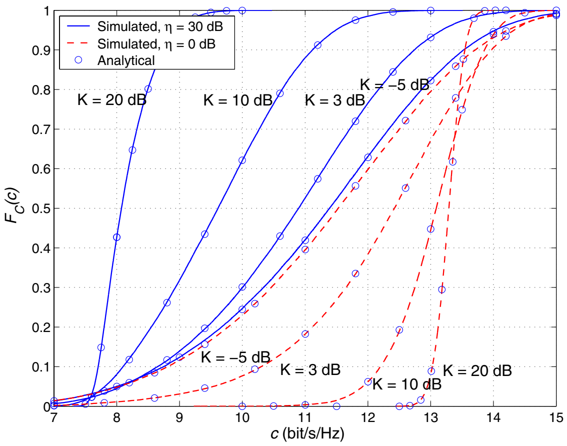

Fig. 1. Capacity CDFs for different Ricean K factors and deviation factors (SNR = 20 dB, $M = N = 2$).

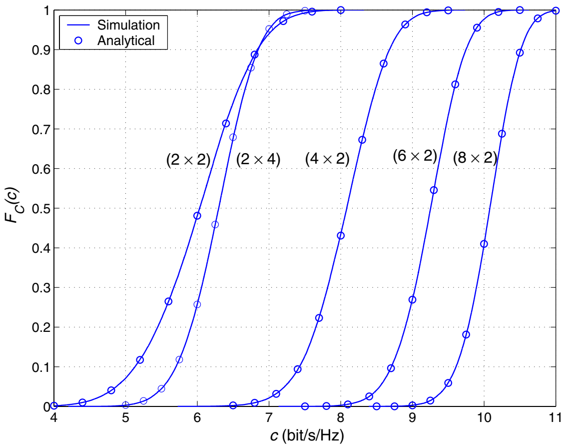

Fig. 2. Capacity distribution for a MIMO system with different dimensions (M × N) (SNR = 10 dB, K = 10 dB and η = 3 dB).

5. CONCLUSIONS AND FURTHER WORK

Analytical expressions for the capacity PDF and CDF are derived for a dual MIMO system in Ricean fading. The derivation results in two sets of expressions, one that is valid for unequal LOS eigenvalues, and one that is valid for equal LOS eigenvalues. The LOS channel matrix is described by a general model that does not put any restrictions on the rank of the matrix. As an example, the special case of uniform linear arrays are studied, and associated LOS eigenvalues are derived. The results show that the simulations follow the analytical results perfectly.

The authors are currently working on capacity expressions for dual systems when the channel is known at the transmitter. In this situation the optimal power allocation scheme is not equal power transmission, as used in this paper, but power allocation done according to the waterfilling scheme.

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