Effects of Finite Coefficient Word Length on Channel Estimator Performance

Asghar Havashki #1, Lars Lundheim #2, Per Gunnar Kjeldsberg #3, Geir E. Øien #4

Department of Electronics and Telecommunications
Norwegian University of Science and Technology
NO-7491, Trondheim, Norway
{ [1]havashki, [2]lundheim, [3]Per.Gunnar.Kjeldsberg, [4]Geir.Oien }@iet.ntnu.no

***Abstract*—In this paper we investigate the impact of finite coefficient word length on channel estimator performance. A theoretical analysis of the increase in channel estimation error due to quantization of estimator coefficients is performed, and by using simulation results, the behavior of this error in different fading environments and for different filter orders is studied. Comparing the simulated results with a theoretical model we have shown that there is a relatively good agreement between results based on the theoretical model and the simulated results. We have shown that there is a closer match between theoretical and simulated results when the input signal is less correlated. Our model is restricted to a flat-fading channel with varying Doppler, and linear FIR estimators.**

I. Introduction

In wireless communications, the power consumption associated with the actual transmission has traditionally been assumed to be dominant in the overall power budget. However, taking digital signal processing (DSP) and RF circuitry power consumption into account becomes increasingly relevant as communication distance decreases [1]. Therefore, co-optimization of circuitry and transmission schemes, and energy efficient DSP design, has received much attention recently.

Orthogonal Frequency Division Multiplexing (OFDM) is one example of a technique which in recent years has become widely applied in wireless communication systems, due to its high data rate transmission capability and high bandwidth efficiency [2]. However, the performance of OFDM and other spectrally efficient schemes depends, among other things, on advanced digital signal processing (DSP) and on the use of efficient and possibly adaptive resource allocation and transmission techniques. These in turn require that *accurate estimates of the channel* are available in the receiver and transmitter. The importance of accurate channel estimation is also demonstrated in [3], a study on the impact of channel estimation error on the performance of linear FIR equalizers. It shows that as $\frac{E_s}{N_0}$ approaches 20 dB and beyond, the influence of the channel estimation error on the overall error-rate becomes important for the scenario discussed. For lower $\frac{E_s}{N_0}$, the additive channel noise dominates the overall error-rate. In this paper we are concerned with wireless communication scenarios where the system performance is primarily limited by available estimation accuracy.

Accurate channel estimation of a time and frequency dispersive wireless fading channel calls for complex estimators, which might lead to significant energy dissipation in such devices. As power has become a major constraint in design of e.g., wireless and sensor ad hoc networks, saving power by reducing the number of bits when realizing DSP operations is an attractive approach. Our aim in this paper is to see how such reduction, when implementing a given channel estimator, impacts the estimation accuracy. Here we shall assume, as a representative example, that a certain type of linear FIR estimator is to be used. When realizing such estimators three quantization effects due to finite word length would be fallen into consideration. Quantization of input data, quantization of filter coefficients, and quantization of the results of the arithmetic operations within the filter. The impact of the mentioned quantization effects on the output of digital filters has been investigated by many researchers. A review of some analytical methods used to describe the impact of these quantization effects on the accuracy of digital filters is given in [4]. In [5] the impact of the mentioned quantization effects on the output, for a direct-form FIR filter, has been analyzed. The authors in [5] present a statistical model for the error variance in the frequency response of the filter due to coefficient quantization. This model is based on the assumption that errors due to the quantization of different coefficients are statistically independent, and that each error is uniformly distributed between $-\frac{\triangle}{2}$ and $\frac{\triangle}{2}$ and thus has zero mean and variance $\frac{\triangle^2}{12}$, where $\triangle$ is the quantization step.

In this paper we investigate the effects of finite filter coefficient word length on channel estimator performance. We first characterize the channel estimation error due to quantization of estimator filter coefficients, and then, by using simulation results, we discuss the behavior of this error. We will study the behavior of the quantization error in different fading environments and for different estimator orders. By comparing the simulated results with a theoretical model we examine the agreement between theoretical and experimental results, under the mentioned conditions.

II. System Model

In our system model (Figure 1) a stream of complex-valued symbols, $s(n)$, with deterministic pilot symbols equally

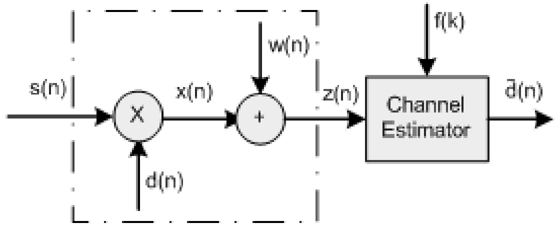

Fig. 1. System model.

spaced within the input stream with period L, enters a time varying Wide Sense Stationary (WSS) flat fading channel. This channel can, for example, represent one subchannel in an OFDM system [1]. The channel estimates are derived from only the pilot symbols, $s(nL)$, which for simplicity are all assumed to be of equal power, with $s(nL) = 1$.

In the channel, the symbols are subjected to fading, with $d(n)$ denoting the complex fading gain. A complex-valued Additive White Gaussian Noise (AWGN) $w(n)$ is then added to the faded signal $x(n)$, so that the resulting received signal $z(n)$ entering the channel estimator becomes:

$$z(n) = s(n) \cdot d(n) + w(n). \tag{1}$$

Here we shall assume that a linear FIR estimator is to be used; it is well known from estimation theory that this is MMSE optimal if the channel exhibits Rayleigh fading. To estimate the channel gain $d(n)$, we employ a non-causal lowpass filter which uses a number of the received pilot symbols, both from the past and future:

$$\hat{d}(n) = \sum_{k=-N/2}^{N/2} f(k) z(n-k). \tag{2}$$

The *optimal* filter coefficient vector in the *maximum a posteriori* (MAP) or minimum MSE sense, $\mathbf{f}_{\text{MAP}} = [f(\frac{-N}{2}), ..., f(\frac{N}{2})]_{\text{MAP}}$, for such an estimator of order N, is given by [6]:

$$\mathbf{f}_{\text{MAP}}^T = \mathbf{r}^T (\mathbf{R} + \frac{\mathbf{1}}{\bar{\gamma}} \mathbf{I})^{-\mathbf{1}} \tag{3}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix, $\mathbf{R}$ is a normalized $N \times N$ covariance matrix (a symmetric Toeplitz matrix), $\mathbf{r}$ is a normalized $1 \times N$ covariance vector, and $\bar{\gamma}$ is the average Channel Signal-to-Noise Ratio (CSNR). By assuming *isotropic scattering* for the fading environment the *Jakes* fading model can be applied [6], which simplifies the computation of $\mathbf{r}$ and $\mathbf{R}$. We have used the Jakes model in our simulations, but this is not a critical limitation of our work.

The channel estimation error, $e(n)$, is now computed as

$$e(n) = d(n) - \hat{d}(n). \tag{4}$$

[1]Of course, in that particular case correlation in frequency could also be exploited for improved channel estimation accuracy.

III. Estimator Filter Coefficient Quantization

We let $\hat{d}(n)$ in (2) be the estimation of the channel gain for the infinite-precision implementation case, with $\sigma_e^2 = \text{Var}[e(n)]$ being the variance of the error $e(n)$ (4). Similarly, if we let $\hat{d}_q(n)$ be the estimated channel gain for the case with *quantized* estimator coefficients, we can define the additional channel estimation error due to coefficient quantization as

$$e_q = \hat{d}_q(n) - \hat{d}(n), \tag{5}$$

The variance of e_q is denoted $\sigma_{e_q}^2$. Now $\hat{d}_q(n)$ becomes:

$$\hat{d}_q(n) = d(n) + e(n) + e_q(n) \tag{6}$$

Assuming $e_q(n)$ and $e(n)$ to be independent, we have:

$$\text{Var}[\hat{d}_q(n) - d(n)] = \text{Var}[e(n) + e_q(n)] = \sigma_e^2 + \sigma_{e_q}^2 \tag{7}$$

A. Equivalent system model

Quantizing the infinite-precision coefficients $f(k)$ in (2) will cause a quantization error sequence $q(k)$; i.e., the quantized coefficients $f_q(k)$ are given as

$$f_q(k) = f(k) + q(k). \tag{8}$$

Substituting the infinite-precision coefficients $f(k)$ in (2) with $f_q(k)$ we arrive at the equivalent finite-precision system model:

$$\begin{aligned} \hat{d}_q(n) &= \sum_{k=-N/2}^{N/2} f_q(k) z(n-k) \\ &= \sum_{k=-N/2}^{N/2} f(k) z(n-k) + \sum_{k=-N/2}^{N/2} q(k) z(n-k). \end{aligned} \tag{9}$$

The last term in (9) is the effective channel estimation error due to quantization of estimator coefficients, $e_q(n)$.This term can be interpreted as the output of an *error filter*, with impulse response $q(k)$, in parallel with the ideal filter $f(k)$. Now, the variance of e_q is given by:

$$\sigma_{e_q}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mid Q(\omega) \mid^2 \cdot \Gamma_{zz}(\omega) \, d\omega \tag{10}$$

where $Q(\omega)$ is the frequency response of e_q, and $\Gamma_{zz}(\omega)$ is the power spectral density of the input signal $z(n)$.

Furthermore, the maximum error in a coefficient value is bounded by:

$$-\frac{\triangle}{2} \le q(k) \le \frac{\triangle}{2} \tag{11}$$

where $\triangle = 2^{-(b-1)}$, and b is the number of bits.

If we also assume that the coefficient error sequence $q(k)$ is uniformly distributed, then:

$$\sum_{k=-N/2}^{N/2} q^2(k) \approx N \cdot \frac{\triangle^2}{12}, \tag{12}$$

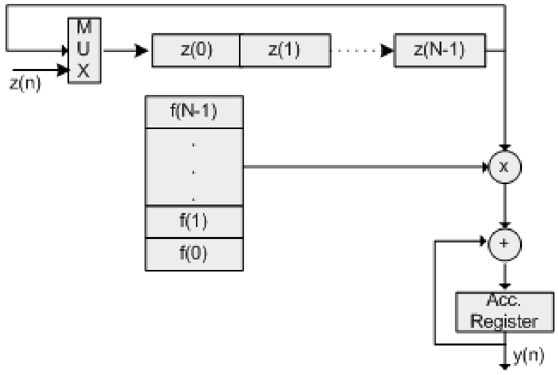

Fig. 2. Channel estimator architecture.

and by Parseval's relation we find:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid Q(\omega) \mid^2 \, d\omega = \overline{\mid Q(\omega) \mid^2} \approx N \cdot \frac{\triangle^2}{12}. \tag{13}$$

To simplify the analysis we assume, as in [7], that the fading process is perfectly bandlimited to the normalized angular frequency interval $[-\omega_D, \omega_D]$, where $\omega_D = \frac{2\pi f_D}{F_s}$, f_D is the maximum Doppler frequency, and F_s the sampling frequency. Then, the power spectral density of the fading process is modeled as [7]:

$$\Gamma_{dd}(\omega) = \begin{cases} \frac{\pi}{\omega_D}\sigma_d^2 & \mid \omega \mid < \omega_D \\ 0 & \text{elsewhere} \end{cases} \tag{14}$$

Now $\sigma_{e_q}^2$ in (10) becomes:

$$\begin{aligned} \sigma_{e_q}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mid Q(w) \mid^2 \cdot (\Gamma_{dd}(\omega) + \Gamma_{ww}(\omega))\, d\omega \\ &= \frac{\sigma_d^2}{2\ \omega_D}\int_{-\omega_D}^{w_D} \mid Q(\omega) \mid^2 \, d\omega + \frac{\sigma_w^2}{2\pi}\int_{-\pi}^{\pi} \mid Q(\omega) \mid^2 \, d\omega \\ &= \sigma_d^2 \cdot \overline{\mid Q_D(\omega) \mid^2} + \sigma_w^2 \cdot \overline{\mid Q(\omega) \mid^2} \end{aligned} \tag{15}$$

where

$$\overline{\mid Q_D(\omega) \mid^2} = \frac{1}{2\ \omega_D}\int_{-\omega_D}^{\omega_D} \mid Q(\omega) \mid^2 \, d\omega,$$

is the mean of $\mid Q_D(\omega) \mid^2$ over the interval $[-\omega_D, \omega_D]$.

If we assume $\overline{\mid Q_D(\omega) \mid^2} = \overline{\mid Q(\omega) \mid^2}$, then by using (13) we get:

$$\sigma_{e_q}^2 = N\frac{\triangle^2}{12}(\sigma_d^2 + \sigma_w^2). \tag{16}$$

IV. Results and Discussion

The architecture used for the real part of the estimator is shown in Figure 2. An identical architecture is used for the imaginary part. Coefficients are saved in a register f, while the sequence of input data being operated on to produce one output sample is saved in a register denoted as z. The results are saved in the accumulator register. To study finite coefficient word length effects on the channel estimator performance, we reduce the number of bits used to hold the coefficients $f(k)$, when running simulations for each case

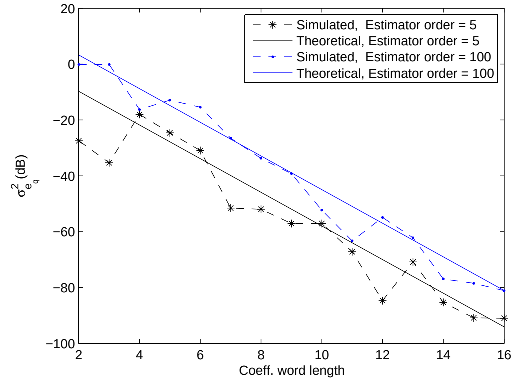

Fig. 3. Simulated and theoretical $\sigma_{e_q}^2$ as function of coefficient word length for estimator order of 5 and 100. Maximum Doppler frequency, f_D, is 200 Hz, sampling frequency, F_s, is 40 kHz, channel SNR is 20 dB and results are for 1000000 simulations. Quantization technique used is SC_RND_ZERO.

Simulations are performed based on (5), and an estimate of the variance of $e(n)$ is computed based on:

$$\hat{\sigma}_e^2 = \frac{1}{L}\sum_{n=1}^{L} \mid e(n) \mid^2, \tag{17}$$

where L is the simulation length.

Simulations are run in the Systemc environment [8] which offers fixed point data types that can be used to accurately model hardware. An object of fixed point type is declared in Systemc as: $sc\_fixed < wl, iwl, q\_mode, o\_mode > x$, where *wl* is the total word length used for fixed point representation, *iwl* is the integer word length which specifies the number of bits to the left of the binary point (.) in a fixed point number, *q_mode* is the quantization mode used, and *o_mode* is the overflow mode. The quantization mode chosen in our simulations is *SC_RND_ZERO*, which rounds the value to the closest representable number if the two nearest representable numbers are not an equal distance apart. This is accomplished by adding the MSB of the removed bits to the remaining bits, otherwise rounding to zero will be performed. For positive numbers this means that the redundant bits are simply deleted, while for negative numbers the MSB of deleted bits are added to the remaining bits. For the overflow mode, *o_mode*, the *SC_SAT* method is chosen, converting the specified value to *MAX* in the case of overflow and to *MIN* in the case when an underflow occurs. The maximum and minimum values (*MAX* and *MIN* respectively) will be determined from the number of bits available.

Figure 3 shows the simulated results, along with theoretical curves given by (16), for an estimator order of 5 and 100, respectively, when the maximum Doppler frequency is $f_D = 200$ Hz (corresponding to a mobile speed of 30 m/s). Figure 4 shows similar results for the case when the maximum

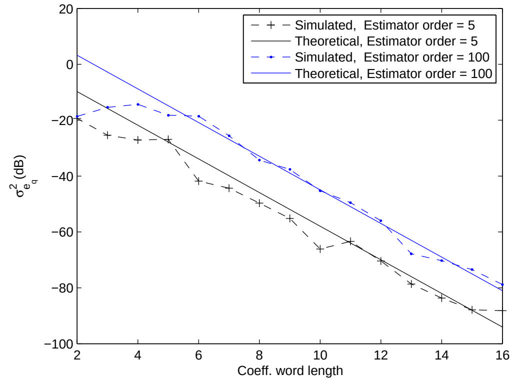

Fig. 4. Simulated and theoretical $\sigma^2_{e_q}$ as function of coefficient word length for estimator order of 5 and 100. Maximum Doppler frequency, f_D, is 2000 Hz, sampling frequency, F_s, is 40 kHz, channel SNR is 20 dB and results are for 1000000 simulations. Quantization technique used is SC_RND_ZERO.

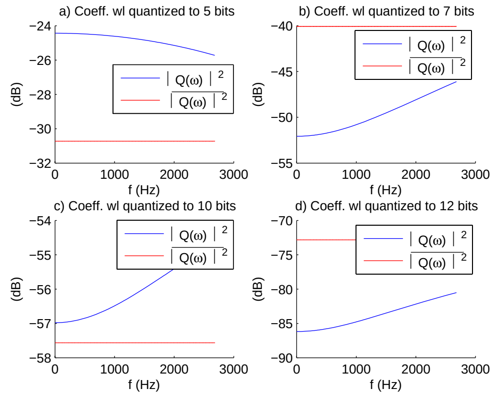

Fig. 5. $\mid Q(\omega) \mid^2$ and $\overline{\mid Q(\omega) \mid^2}$ for the cases where rounding method is used to quantize the coefficients to 5, 7, 10 and 12 bits The estimator order is 5, sampling frequency, F_s, is 40 kHz and maximum Doppler frequency is $f_d = 200$ Hz.

Doppler frequency is $f_D = 2000$ Hz (corresponding to a mobile speed of 300 m/s), again for estimator order of 5 and 100, respectively.

Deviations from the theoretical curve have two reasons. Firstly, the relation (12) is never exact, in particular for large values of $\triangle$, and is mainly responsible for deviations at small values of b. Secondly, the assumption $\overline{\mid Q_D(\omega) \mid^2} = \overline{\mid Q(\omega) \mid^2}$ is also only an approximation. This is well illustrated by figure 5, where we see e.g. that for $b = 10$, $\overline{\mid Q_D(\omega) \mid^2} \approx \overline{\mid Q(\omega) \mid^2}$. Then the assumption is accurate, and we observe a correspondingly good match between the curves of Figure 3. When $\overline{\mid Q_D(\omega) \mid^2}$ is obviously above $\overline{\mid Q(\omega) \mid^2}$, such as in Figure 5 a), the theoretical value of $\sigma^2_{e_q}$ will be too low, which is again confirmed by Figure 3.

Furthermore, looking at the results we observe a better match between the simulated and theoretical plots for the results in Figure 4 than for the results in Figure 3. The applied maximum Doppler frequency in the former case, where the fading input is less correlated, is 10 times higher than the maximum Doppler frequency in the later one. Hence, $\overline{\mid Q_D(\omega) \mid^2}$ is computed over a larger interval, and expected to be closer to $\overline{\mid Q(\omega) \mid^2}$.

In general, in the cases where the mean of $\mid Q(w) \mid^2$ over the interval $[-\omega_D, \omega_D]$ is close to the mean of $\mid Q(w) \mid^2$ over the interval $[-\pi, \pi]$, i.e., $\overline{\mid Q_D(\omega) \mid^2} = \overline{\mid Q(\omega) \mid^2}$, we experience a close match between (16), the theoretical model, and (15), therefore experiencing a good match between the theoretical and simulated results. This assumption becomes more accurate the lager the maximum Doppler frequency becomes.

But despite the observed deviations, based on the results, we can conclude that the error variance represented by the theoretical case behaves as a good approximation for the error variance of the quantization error when rounding is performed to quantize the coefficients.

V. Conclusions

In this paper we have investigated the impact of finite coefficient word length on channel estimator Performance. In our study, we have assumed that errors due to the quantization of different coefficients are statistically independent, and that each error is uniformly distributed between $-\frac{\triangle}{2}$ and $\frac{\triangle}{2}$ and thus has zero mean and variance $\frac{\triangle^2}{12}$, where $\triangle$ is the quantization step. Based on the given assumption, we have performed a theoretical analysis of the increase in channel estimation error due to quantization of estimator coefficients. Then, by using simulation results, we have studied the behavior of this error in different fading environments and for different filter orders. By comparing the simulated results with a theoretical model we have shown that there is a relatively good agreement between results based on the theoretical model and the simulated results. We have shown that there is a closer match between theoretical and simulated results when the input signal is less correlated. We have restricted our model to a flat-fading channel with varying Doppler, and linear FIR estimators.

References

[1] A. Goldsmith, *Wireless Communications*, 1st ed. Cambridge University Press, 2005.

[2] X. Wang and H. V. Poor, *Wireless Communication Systems: Advanced Techniques for Signal Reception*, 1st ed. Prentice Hall, 2004.

[3] D. Welden, F. Simoens, H. Steendam, and M. Moeneclaey, “Impact of channel estimation errors on the performance of linear FIR equalizers for frequency selective MIMO channels,” in *Proc. SCVT’05*, Enschede, The Netherlands, Nov. 2005.

[4] B. Liu, “Effect of finite word length on the accuracy of digital filters,” *IEEE Transactions on Circuits Theory*, vol. 18, no. 6, pp. 670– 677, Nov. 1971.

[5] D. S. K. Chan and L. R. Rabiner, “Analysis of quantization errors in the direct form for finite impulse response digital filters,” vol. AU-21, no. 4, Aug. 1973, pp. 354– 366.

[6] H. Holm, “Adaptive coded modulation performance and channel estimation tools for flat fading channels,” Ph.D. dissertation, Norwegian University of Science and Technology, 2002, available at: www.henrikholm.com/avhandling/thesis.pdf.

[7] G. E. ien and K. J. Hole, “Maximum average spectral efficiency of slowly varying rayleigh fading channels with pilot-symbol-assisted channel estimation,” in *in Proc. European Personal Mobile Communications*, Vienna, Austria.

[8] Systemc 2.0.1 language reference manual. [Online]. Available: www.systemc.org.