# On the Limiting Behavior of Directional MIMO Channels

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*Abstract* — **An asymptotic analysis (in the number of antennas) of the achievable transmission rate on directional based models is conducted using tools from random matrix theory. A central limit theorem is provided on the asymptotic behavior of the mutual information and validated in the finite case by simulations. The results are extremely useful in terms of designing a system based on criteria such as quality of service or in optimizing transmissions in multiuser networks.**

# I Introduction

In this contribution[1], we derive the asymptotic distribution of the mutual information considering single directional models: Directions of Arrival (DoA) and Directions of Departure (DoD) based models. It is shown in particular that if $\mathbf{H}_{r\times t}$ is a $r \times t$ ($r$ and $t$ are respectively the number of receiving and transmitting antennas) DoA channel matrix (or DoD channel matrix), then the following result holds:

$$\lim_{t\to\infty, \frac{r}{t}=\beta} I^M(t,r,\rho) - t\mu \rightarrow N(0,\sigma^2) \quad (1)$$

$I^M(t,r,\rho) = \log_2 \det\left(\mathbf{I}_t + \frac{\rho}{t}\mathbf{H}^H\mathbf{H}\right)$ ($\rho$ is the received SNR) is the mutual information with Gaussian input entries and covariance $\mathbf{Q} = \mathbf{I}$. The convergence is in distribution. Only the mean $\mu$ and the variance $\sigma^2$ are needed to fully characterize the distribution [2]. The result 1 generalizes the work of Telatar ([1])and Müller [2] and has already been proved by Kamath et al. [3] in the i.i.d Gaussian channel case. For proving the theorem, results of random matrix theory will be used [4]. One of the useful features of random matrix theory is the ability to predict, under certain conditions, the behavior of the empirical eigenvalue distribution of products or sums of matrices. The results are striking in terms of closeness to simulations with reasonable matrix size and enable us to derive linear spectral statistics for these matrices with only few meaningful parameters. For example, let $q$ denote the outage probability and ${I_q}^M$ the corresponding outage mutual information, then:

$$\begin{aligned} q &= \int_{-\infty}^{{I_q}^M} \mathrm{d}I^M p(I^M) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{{I_q}^M} \mathrm{d}I^M e^{-\frac{(I^M - t\mu)^2}{2\sigma^2}} \\ &= 1 - Q\left(\frac{{I_q}^M - t\mu}{\sigma}\right) \end{aligned}$$

$${I_q}^M = t\mu + \sigma Q^{-1}(1-q)$$

We define $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \mathrm{d}t e^{\frac{-t^2}{2}}$. The outage mutual information depends therefore only on two parameters: $\mu$ and $\sigma$. The result is also useful in multiuser networks. From a practical point of view, the receiver can compute the theoretical mean and the variance of the mutual information after estimation of the channel characteristics (steering directions,...). This information is then sent back to the transmitter to optimize the scheduling of the network [3]. One interesting point of the feedback mechanism is that also only two values (the mean and the variance) are needed. This reduces drastically the overhead of feedback transmissions.

# II DoA based model

## A Some considerations

### A.1 Model

The model under consideration is a DoA based model which has the following form:

$$\mathbf{H} = \frac{1}{\sqrt{s}} \begin{pmatrix} e^{j\phi_{1,1}} & \dots & e^{j\phi_{1,s}} \\ \vdots & \ddots & \vdots \\ e^{j\phi_{r,1}} & \dots & e^{j\phi_{r,s}} \end{pmatrix} \begin{pmatrix} P_1^r & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & P_s^r \end{pmatrix} \mathbf{\Omega}_{s\times t}$$

$\Phi$ ($\mathbf{H} = \frac{1}{\sqrt{s}}\Phi\mathbf{P^r}\mathbf{\Omega}$) is a $r \times s$ matrix ($s$ is the number of scatterers) which represents the directions of arrival from randomly positioned scatterers to the receiving antennas. $P_i^r$ are the amplitudes of the different steering vectors. $\mathbf{\Omega}_{s\times t}$ is an $s \times t$ i.i.d Gaussian matrix which represents the scattering environment between the transmitting antennas and the scatterers (see figure (1)). This model is a single directional (only the effect of the DoA are analyzed) version of the double directional model developed in [6] and was proved to be consistent within the maximum entropy framework.

### A.2 Mutual information

We are interested in the behavior of $I^M(t,r,s,\rho) = \log_2\det\left(\mathbf{I}_t + \frac{\rho}{t}\mathbf{H}^H\mathbf{H}\right)$ and in particular the eigenvalue distribution of $\frac{1}{t}\mathbf{H}^H\mathbf{H} = \frac{1}{ts}\mathbf{\Omega}_{s\times t}^H\Phi_{r\times s}^H\Phi_{r\times s}\mathbf{\Omega}_{s\times t}$. Let us first make some assumptions on the matrix of directions of arrival.

**Assumption:** As the matrix size $\frac{1}{s}\mathbf{P^r}^H\Phi_{r\times s}^H\Phi_{r\times s}\mathbf{P^r}$ grows large with $\gamma = \frac{r}{s}$ remaining fixed, the empirical

[1] This work is part of the European FLOWS project.
[2] Note that $\mu = \mu(\beta,\rho)$ and $\sigma^2 = \sigma^2(\beta,\rho)$ depend on $\beta = \frac{r}{t}$ and $\rho$.
[3] Some results on the capacity of a MIMO multi-user network (where all the users have different angles of arrival) in the large system limit (high number of antennas) can be found in [5].

eigenvalue $G_{s,r}$ converges in distribution to a fixed $G_{\mathrm{doa}}$[4]: $G_{s,r}(\lambda) = \frac{1}{s} \mid \{j : \lambda_j \leq \lambda\} \mid \rightarrow G_{\mathrm{doa}}(\lambda)$.

In this case, the following theorem holds:

**Theorem 1** *[4] As $t \rightarrow \infty$ with $s = \xi t, I^M{}_{doa}(t, r, s, \rho) - t\mu$ converges in distribution to a $N(0, \sigma^2)$ random variable where:*

$$\mu = \int_0^\infty ln(1 + \rho\lambda) dF(\lambda)$$

$$m(z) = \int \frac{dF(\lambda)}{\lambda - z}$$

$$z = \frac{-1}{m(z)} + \xi \int \frac{x}{1 + m(z)x} dG_{doa}(x)$$

$$\sigma^2 = -\frac{1}{4\pi^2} \int_{C_x} \int_{C_y} \frac{ln(1 + \rho x) ln(1 + \rho y)}{(m(x) - m(y))^2} m'(x) m'(y) dx dy$$

*$C_x$ and $C_y$ are any closed contour that enclose the support of $F$.*

Theorem (1) is extremely important as it shows that only the limiting eigenvalue distribution of the steering directions matters: in other words, two antenna configuration can yield the same throughput as long as they give rise to the same eigenvalue distribution for the steering matrix. Based on this result, a future mobile scenario the authors would like to advocate is the following: imagine a set of reconfigurable antennas that can move on a grid. The antennas are at the beginning displayed in a Uniform Linear Array geometry. Once the transmission starts, the angles of arrival and the distances of the scatterers to the antennas are determined. The position of the antennas (for fixed scatterers) on the grid are then optimized in order to increase mutual information using the previous theorem. This is a viable scenario from a reconfigurable antenna perspective and gives means for future research in the field of antenna design. The antenna design problem can therefore be related to an eigenvalue optimization problem.

# B ULA and fourier directions

In this part, the geometry of the receiving antenna is taken into account: in the case of a uniform linear array and far field approximation, the steering vector has the following form $[1, e^{-j2\pi \frac{d \sin(\phi)}{\lambda}}, ..., e^{-j2\pi \frac{d(r-1)\sin(\phi)}{\lambda}}]$ where $d$ is the antenna spacing and $\phi$ is the direction of arrival. For simulations purpose, we will take $d = \frac{\lambda}{2}$. We will also suppose that $s \leq r$. In order to have tractable explicit formulas, we will analyze the distribution of scatterers in the case where for any $i$ there exists a $k$ such as $\sin(\phi_i) = \frac{2k}{r}$. This case can be seen as an extreme case where all the scatterers are maximally distant from each other (uncorrelated scattering) and is a single directional version of the model developed in [7].

[4]In this representation, the angular spread is taken into account through matrix $\mathbf{P^r}$. As a matter of fact, the notion of angular spread is implicit for the matrix $\frac{1}{s}\mathbf{P^r}^H \Phi^H_{r\times s}\Phi_{r\times s}\mathbf{P^r}$ in order to have a limiting eigenvalue distribution as the number of scatterers increases. The random matrix approach (in terms of asymptotic analysis) is therefore in this case even more appealing as it takes into account the tremendous number of scatterers within a cluster.

## B.1 Equal power case

In this case, we consider $\mathbf{P^r} = \mathbf{I}_s$. As a consequence, the limiting eigenvalue distribution $G_{\mathrm{doa}}$ of $\frac{1}{s}\Phi^H_{r\times s}\Phi_{r\times s}$ has the following expression (since the column vectors of $\Phi_{r\times s}$ are orthogonal): $G_{\mathrm{doa}}(\lambda) = \delta(\lambda - \gamma)$

**Proposition 1** *In this case, $\mu_{doa}(\xi, \gamma, \rho)$ and $\sigma^2_{doa}(\xi, \gamma, \rho)$ are equal to:*

$$\begin{aligned} \mu_{doa}(\xi, \gamma, \rho) &= \xi \ln(1 + \rho\gamma - \rho\gamma\alpha_{doa}(\xi, \gamma, \rho)) \\ &+ \ln(1 + \rho\xi\gamma - \rho\gamma\alpha_{doa}(\xi, \gamma, \rho)) - \alpha_{doa}(\xi, \gamma, \rho) \end{aligned}$$

$$\sigma^2_{doa}(\xi, \gamma, \rho) = -\ln[1 - \frac{{\alpha_{doa}}^2(\xi, \gamma, \rho)}{\xi}]$$

*with*

$$\alpha_{doa}(\xi, \gamma, \rho) = \frac{1}{2}\left[1 + \xi + \frac{1}{\rho\gamma} - \sqrt{(1 + \xi + \frac{1}{\gamma\rho})^2 - 4\xi}\right]$$

**Proof 1** *The proof can be found in [6]. Note that in the case $s = t$, matrix $\Phi$ is a $r \times r$ Fourier matrix and the channel model is i.i.d Gaussian. In this case, the previous formulas are consistent with [3].*

In figure (2), simulations have been conducted with $r = t = 8$ antennas and an SNR of 10dB. In this case, $\frac{s}{r} = \xi = \frac{1}{\gamma}$. Three cases have been plotted $\xi = \frac{1}{4}$, (2 scatterers), $\xi = \frac{1}{2}$, (4 scatterers) and finally $\xi = 1$, (8 scatterers). A close match between the theoretical formulas and simulations is observed for only 8 antennas.

In figures (3) and (4), the asymptotic mean and asymptotic variance have been simulated versus $\frac{s_1}{t}$ for a system of $32 \times 32$ antennas and compared to the theoretical formulas. A close match between theory and simulations is also obtained in this case. In figure.(3), the asymptotic mean of the mutual information with respect to $\xi$ at 10dB shows that the number of scatterers does not yield a linear gain. The result also acknowledges the well known fact that the full transmission potential is obtained when the number of scatterers is equal to the number of antennas.

## B.2 Non-equal power case

We consider in this case that there is a finite set of $K_r$ distinct powers. In other words, $\frac{s}{K_r}$ receiving steering directions have the same power. As a consequence, the limiting eigenvalue distribution $G_{\mathrm{doa}}$ of $\frac{1}{s}\mathbf{P^r}^H {\Phi_{r\times s}}^H \Phi_{r\times s}\mathbf{P^r}$ has the following expression:

$$G_{\mathrm{doa}}(\lambda) = \frac{1}{K_r} \sum_{i=1}^{K_r} \delta(\lambda - \gamma({P_i}^r)^2)$$

**Proposition 2** *In this case, $\mu_{doa}(\xi, \gamma, \rho)$ and $\sigma^2_{doa}(\xi, \gamma, \rho)$ are equal to:*

$$\begin{aligned} \mu_{doa}(\xi, \gamma, \rho) &= -\ln(\alpha_{doa}) + \frac{\xi}{K_r} \sum_{i=1}^{K_r} \ln(1 + \rho({P_i}^r)^2 \gamma \alpha_{doa}) \\ &- (1 - \alpha_{doa}) \end{aligned}$$

$$\sigma^2_{doa}(\xi,\gamma,\rho) = -\ln((-1)^{K_r+1} + (-1)^{K_r}\frac{\rho^2\xi{\alpha_{doa}}^2}{K_r} \sum_{i=1}^{K_r}\frac{(\gamma({P_i}^r)^2)^2}{(1+\rho\gamma({P_i}^r)^2\alpha_{doa})^2}(-1)^{K_r+1})$$

*with* $\frac{1}{K_r}\sum_{i=1}^{K_r}\frac{1}{1+\rho\gamma({P_i}^r)^2\alpha_{doa}} = \frac{\alpha_{doa}}{\xi} - \frac{1}{\xi} + 1$

**Proof 2** *The proof can be found in [6].*

In figure (5) and figure (6), the asymptotic mean and variance of the mutual information have been plotted versus the amplitude ${P_1}^r$. A close match between theoretical predictions and simulations is obtained for a low number of antennas ($8\times 8$ MIMO system). More importantly, one can observe that the best throughput is obtained when all the steering directions have equal power.

## C Fourier versus random directions

One important question is to know whether, for a given number of scatterers, far field uncorrelated scattering yields better performance than near field scattering. The answer has a direct impact on the understanding and the design of future mobile systems. In the case of far field scattering, one can define angles of arrival and steering directions. In this case, each column of matrix $\Phi$ has a simple expression $[1, e^{-j2\pi\frac{d\sin(\phi)}{\lambda}}, ..., e^{-j2\pi\frac{d(r-1)\sin(\phi)}{\lambda}}]$. However, in the case of near field scattering, the expression is not simple and depends not only on the angles but also on the position. In order to have a tractable formula for the near field effect, we will consider that the exponential entries of matrix $\Phi$ are independent and identically distributed random variables with zero mean and unit variance. The value of the angles do not change during the whole transmission. This is a limiting case of near field scattering (all the rays, for a given scatterer do not come from the same direction)[5]. In this case, the Stieljes transform of the limiting eigenvalue distribution of $\frac{1}{s}\Phi^H\Phi$ is given by [8]:

$$m_{G_{\text{doa}}}(z) = \sqrt{\frac{(1-\gamma)^2}{4z^2} - \frac{(1+\gamma)}{2z} + \frac{1}{4}} - \frac{1}{2} - \frac{(1-\gamma)}{2z}$$

Considering the case $t = r$, the mean mutual information is given by:

$$\mu = \int_0^\rho \left(\frac{-1}{6\gamma\rho^2}\tau(\gamma,\rho) - \frac{2}{3\rho}\frac{\rho\gamma^2 - 2\gamma\rho - 3\gamma + \rho}{\gamma\tau(\gamma,\rho)} + \frac{1}{3}\frac{1+2\gamma}{\gamma\rho}\right)d\rho$$

with $\tau(\gamma,\rho) = ((8\rho\gamma^3 - 24\rho\gamma^2 + 24\rho\gamma + 72\gamma^2 + 36\gamma - 8\rho + 12\sqrt{3}*$

$$\sqrt{\frac{-4\rho^2 + 12\rho^2\gamma - \rho - 12\rho^2\gamma^2 + 4\gamma^3\rho^2 + 8\rho\gamma^2 + 20\gamma\rho + 4\gamma}{\rho}}\gamma)\rho^2)^{\frac{1}{3}}$$

We have plotted in figure (7) the theoretical ergodic mutual information per receiving antenna of the random directions scenario at 10 dB for various ratio of scatterers ($\frac{s}{r}$ ranges from 0 to 1). We have also plotted a simulated curve with a system of $8\times 8$ antennas. A close match between the theoretical formula and the simulations is obtained. We have also plotted the asymptotic mean mutual information of the far field ULA scenario where the scatterers are uncorrelated and given by Fourier directions (see section. B). One can observe that far field scattering on Fourier directions yields better performance than near field scattering. One of the conclusions of this observation is that a better transmission occurs when the mobile is far from the scatterers and the scatterers are located in distant positions.

[5]We agree on the fact that the near-field case is more complicated as the phases are not completely independent but linked through the geometry of the antenna. We mainly use the random approach in order to have tractable mutual information formulas.

# III DoD based model

## A Some considerations

### A.1 Model

The model under consideration in the DoD case (see figure.(8)) is:

$$\mathbf{H} = \frac{1}{\sqrt{s_1}}\mathbf{\Omega}_{r\times s_1}\begin{pmatrix} P_1^t & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & P_{s_1}^t \end{pmatrix}\begin{pmatrix} e^{j\psi_{1,1}} & \dots & e^{j\psi_{1,t}} \\ \vdots & \ddots & \vdots \\ e^{j\psi_{s_1,1}} & \dots & e^{j\psi_{s_1,t}} \end{pmatrix}$$

$\Psi$ ($\mathbf{H} = \frac{1}{s_1}\mathbf{\Omega}_{r\times s_1}\mathbf{P^t}\Phi_{s_1\times t}$) is a $s_1\times t$ matrix ($s_1$ is the number of scatterers) which represents the directions of departure from the transmitting antennas to randomly positioned scatterers. ${P_i}^t$ are the amplitudes of the different steering directions. $\mathbf{\Omega}_{r\times s_1}$ is an $r\times s_1$ i.i.d Gaussian matrix which represents the scattering environment between the receiving antennas and the scatterers.

### A.2 Mutual information

We are interested in the behavior of

$$\begin{aligned} I^M(t,r,s_1,\rho) &= \log_2\det\left(\mathbf{I}_t + \frac{\rho}{t}\mathbf{H}^H\mathbf{H}\right) \\ &= \log_2\det\left(\mathbf{I}_r + \frac{\rho}{t}\mathbf{H}\mathbf{H}^H\right) \\ &= \log_2\det\left(\mathbf{I}_r + \frac{\rho\frac{r}{t}}{r}\mathbf{H}\mathbf{H}^H\right) \end{aligned}$$

Therefore, only the eigenvalue distribution of

$$\frac{1}{r}\mathbf{H}\mathbf{H}^H = \frac{1}{rs_1}\mathbf{\Omega}_{r\times s_1}\mathbf{P^t}\Psi_{s_1\times t}\Psi^H_{s_1\times t}\mathbf{P^t}^H\mathbf{\Omega}^H_{r\times s_1}$$

is of interest. As before, some assumptions on the matrix of the directions of departure are to be made.

**Assumption:** The matrix size $\frac{1}{s_1}\mathbf{P^t}\Psi_{s_1\times t}\Psi^H_{s_1\times t}\mathbf{P^t}^H$ grows large with $\xi_1 = \frac{s_1}{t}$ remaining fixed such that the empirical eigenvalue $G_{s_1,t}$ converges in distribution to a fixed $G_{\text{dod}}$

$$G_{s_1,t}(\lambda) = \frac{1}{s_1}\mid\{j : \lambda_j \le \lambda\}\mid \rightarrow G_{\text{dod}}(\lambda)$$

In this case, the following result holds:

**Theorem 2** *[4] As $r \to \infty$ with $r = \gamma_1 s_1$, $I^M{}_{dod}(t, r, s_1, \rho) - t\mu_{dod}(\xi_1, \gamma_1, \rho)$ converges in distribution to a $N(0, \sigma^2{}_{dod})$ random variable where:*

$$\mu_{dod}(\xi_1, \gamma_1, , \rho) = \gamma_1\xi_1 \int_0^{\infty} \ln(1 + \rho\xi_1\gamma_1\lambda) dF_{dod}(\lambda)$$

$$m_{f_{dod}}(z) = \int \frac{dF_{dod}(\lambda)}{\lambda - z}$$

$$z = \frac{-1}{m_{f_{dod}}(z)} + \frac{1}{\gamma_1} \int \frac{x}{1 + m_{f_{dod}}(z)x} dG_{dod}(x)$$

$$\sigma^2{}_{doa} = \frac{-1}{4\pi^2} \int_{C_x} \int_{C_y} \frac{\ln(1 + \rho\gamma_1\xi_1 x)\ln(1 + \rho\gamma_1\xi_1 y)}{(m_{f_{dod}}(x) - m_{f_{dod}}(y))^2} m'_{f_{dod}}(x) m'_{f_{dod}}(y) dx dy$$

*$C_x$ and $C_y$ are any closed contour that enclose the support of $F$.*

## B ULA and fourier directions

In this case, the geometry of the antennas is taken into account: in the case of a ULA antenna and far field approximation, each line of matrix $\Psi$ has the following expression: $[1, \ldots, e^{j2\pi \frac{d(t-1)\sin(\psi_i)}{\lambda}}]$. For simulations purpose, we will take $d = \frac{\lambda}{2}$. We will also suppose that $s_1 \leq t$. In order to have a tractable formula, we will analyze the distribution of scatterers in the case where for any $i, \exists k$ such as $\sin(\psi_i) = \frac{2k}{t}$.

### B.1 Equal power case

We suppose that $\mathbf{P^t} = \mathbf{I}_{s_1}$. As a consequence, the limiting eigenvalue distribution $G_{\text{dod}}$ has the following expression:

$$G_{\text{dod}}(\lambda) = \delta(\lambda - \frac{1}{\xi_1})$$

**Proposition 3** *In this case, $\mu_{dod}(\xi_1, \gamma_1, \rho)$ and $\sigma^2_{dod}(\xi_1, \gamma_1, \rho)$ are equal to:*

$$\begin{aligned}
\mu_{dod}(\xi_1, \gamma_1, \rho) &= \xi_1 \ln(1 + \rho\gamma_1 - \rho\alpha_{dod}(\xi_1, \gamma_1, \rho)) \\
&+ \gamma_1\xi_1 \ln(1 + \rho - \rho\alpha_{dod}(\xi_1, \gamma_1, \rho)) \\
&- \xi_1 \alpha_{dod}(\xi_1, \gamma_1, \rho)
\end{aligned}$$

$\sigma_{dod}{}^2(\xi_1, \gamma_1, \rho) = -\ln[1 - \frac{\alpha_{dod}{}^2(\xi_1, \gamma_1, \rho)}{\gamma_1}]$
*with*

$$\alpha_{dod}(\xi_1, \gamma_1, \rho) = \frac{1}{2}[1 + \gamma_1 + \frac{1}{\rho} - \sqrt{(1 + \gamma_1 + \frac{1}{\rho})^2 - 4\gamma_1}]$$

**Proof 3** *The proof can be found in [6].*

In figure (9), simulations have been conducted in the case where $r = 4$ and $t = 8$ [6]. We have plotted the cumulative mutual information for $\xi_1 = 0.25$, $\xi_1 = 0.5$ and $\xi_1 = 1$ ($\xi_1 = \frac{s_1}{t}$) at 10dB . Here again, a close match between the theoretical curve and simulations (with quite a small number of antennas) is obtained. In figure (10), we have plotted the asymptotic mutual information in a system such as $\frac{r}{t} = \frac{1}{2}$ (less antennas at the receiving side) versus the ratio $\frac{s}{t}$ with the DoA and DoD model in the case of Fourier directions. For a given number of scatterers $s$, the antennas located at the receiving side provide a better throughput then the ones located at the transmitting side.

### B.2 Non-equal power case

We consider in this case that there is a finite set of $K_t$ distinct powers. In other words, $\frac{s_1}{K_t}$ steering directions have the same power. As a consequence, the limiting eigenvalue distribution $G_{\text{dod}}$ of $\frac{1}{s_1}\mathbf{P^t}\Psi_{s_1\times t}{}^H\Psi_{s_1\times t}\mathbf{P^t}$ has the following expression: $G_{\text{dod}}(\lambda) = \frac{1}{K_t}\sum_{i=1}^{K_t} \delta(\lambda - \frac{(P_i{}^t)^2}{\xi_1})$

**Proposition 4** *In this case, $\mu_{dod}(\xi_1, \gamma_1, \rho)$ and $\sigma^2_{dod}(\xi_1, \gamma_1, \rho)$ are equal to:*

$$\begin{aligned}
\mu_{dod}(\xi_1, \gamma_1, \rho) &= \frac{\xi_1}{K_t} \sum_{i=1}^{K_t} \ln(1 + \frac{\rho(P_i{}^t)^2 \alpha_{dod}}{\xi_1}) + \gamma_1\xi_1 \ln(\gamma_1\xi_1) \\
&- \gamma_1\xi_1 \ln(\alpha_{dod}) - (\gamma_1\xi_1 - \alpha_{dod})
\end{aligned}$$

$$\begin{aligned}
\sigma^2_{dod}(\xi_1, \gamma_1, \rho) &= -\ln((1)^{K_t+1} \\
&+ (-1)^{K_t} \frac{\rho^2\gamma_1\alpha_{dod}}{K_t} \sum_{i=1}^{K_t} \frac{(P_i{}^t)^4}{(1 + \frac{\rho(P_i{}^t)^2\alpha_{dod}}{\xi_1})^2})
\end{aligned}$$

*with* $\frac{1}{K_t}\sum_{i=1}^{K_t} \frac{1}{1 + \frac{\rho(P_i{}^t)^2\alpha_{dod}}{\xi_1}} = \frac{\alpha_{dod}}{\xi_1} - \gamma_1 + 1$

**Proof 4** *The proof can be found in [6].*

Notice that in the case $r = t = s_1$, the same results as for the DoA case (see previous section (B.2)) are obtained. Hence, one can observe that the best throughput is obtained when all the steering directions have equal power. This suggests a power loading strategy at the transmitter based only on the power of the steering directions: indeed, if the power of the steering directions are known at the transmitter, than one should equalize the power across the steering directions.

## C Fourier versus random directions

As before, we would like to quantify the impact of near field scattering in the DoD based model. In this case, we will suppose that the exponential entries of matrix $\Psi$ are a realization of independent and uniformly identically distributed variables with zero mean and unit variance. Since the model is the product of two random i.i.d matrix in this case, all the results and conclusions of the previous section (C) on the DoA based model are also valid here.

# IV Conclusion

In this paper, the asymptotic outage mutual information of directional models has been derived and shown to fit simulations with reasonable matrix size. Recently, the results have been extended in order to take into account the

[6] The case $t = r$ has not been considered as it provides the same performance as the direction of arrival based model since $\frac{s}{t} = \xi_1 = \frac{1}{\gamma_1}$.

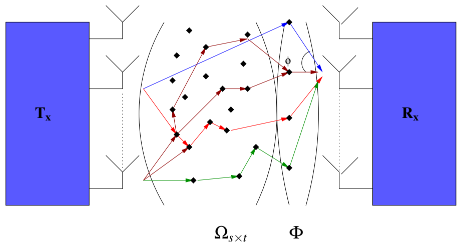


Figure 1: Directions of arrival based model

double directionality as well as the power of the steering directions [9]. Note that the directional models as well as their asymptotic analysis have been validated in a recent measurement campaign performed in Oslo [6].

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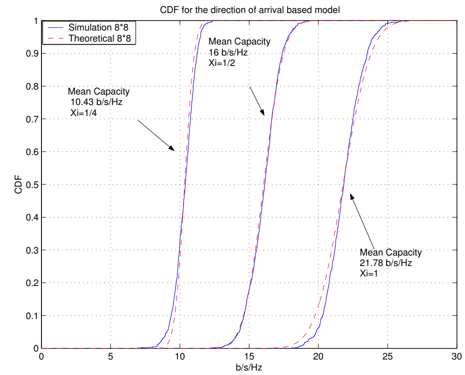


Figure 2: Mutual information cumulative distribution

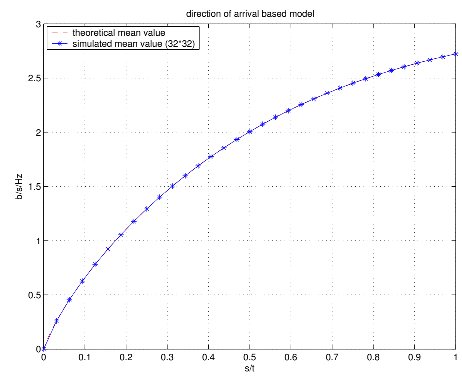


Figure 3: Theoretical versus simulated mean at 10dB

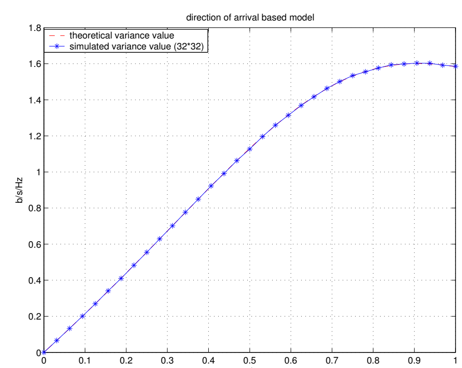


Figure 4: Theoretical versus simulated variance at 10dB

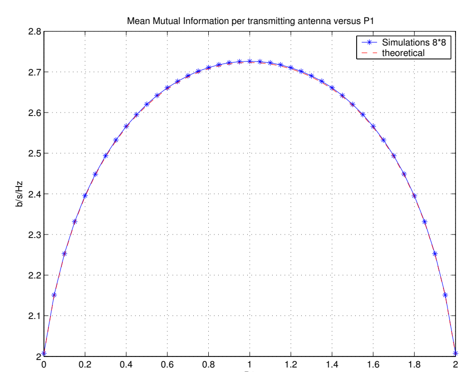


Figure 5: Mean capacity per transmitting antenna versus ${P_1}^r$ at 10dB

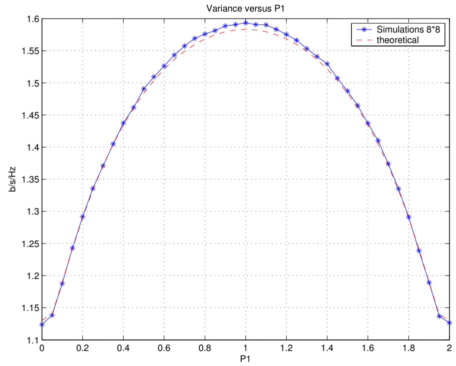


Figure 6: Variance versus ${P_1}^r$ at 10dB

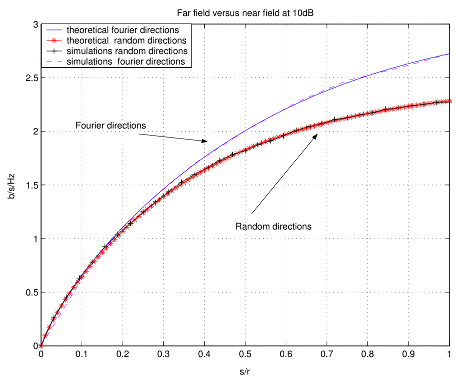


Figure 7: Fourier versus random directions at 10dB

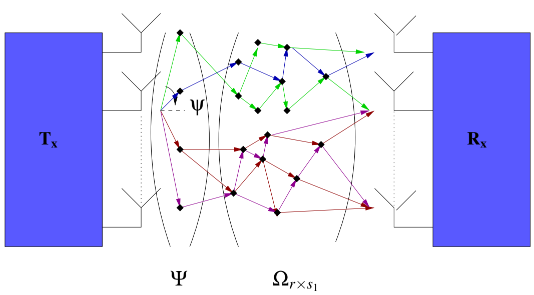


Figure 8: Directions of departure based model

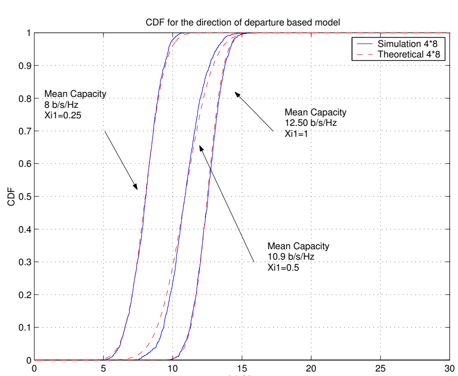


Figure 9: Mutual information cumulative distribution

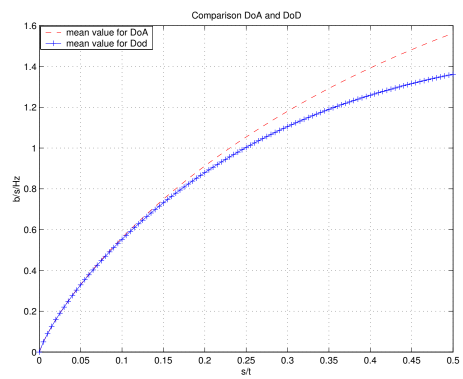


Figure 10: Asymptotic mean mutual information per transmitting antenna versus $\frac{s}{t}$