Minimum Probability of Error and Channel Capacity in Large Dual Antenna Array Systems

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Abstract

Uncoded bit error probability with maximum-likelihood detection and channel capacity is derived analytically for binary signaling on the dual antenna array channel with correlated fading in the large system limit. The example of fading correlation due to poor scattering is analyzed in greater detail. In case of poor scattering (or high load), a waterfall behavior of the uncoded bit error probability is observed.

Index terms — dual antenna arrys, MIMO systems, error probability, replica method, channel capacity, limited scattering

1 Introduction

Communication via dual antenna arrays (i.e. multiple antenna elements at both ends of the link) promise reliable communication at high data rates [1]. The potential performances of such systems were analyzed in literature under various idealized assumptions. Telatar [2] assumed that the channel coefficients are fading statistically independently—an assumption which holds well only if there are much fewer antenna elements than scattering objects. Other theoretical work on the antenna array channel based on the independence assumption includes e.g. [3, 4][1].

In earlier work, the author [5] assumed limited scattering, but a very large number of antenna elements and calculated channel capacity as well as the performance of a linear detector based upon the minimum mean-squared error optimization criterion. Both Telatar and the author assumed Gaussian distributed channel input alphabet. Though it is optimum, Gaussian alphabet cannot be used in practice, but it can only be approximated to a certain extent. In practice, a binary (or quaternary) input alphabet is often used.

This work takes into account statistical dependecies among channel coefficients and calculates channel capacity for binary input alphabet as well as the exact uncoded error probability for maximum likelihood detection in the large antenna limit analytically. The calculations are based upon the replica method developed in statistical physics [6]. It was successfully used to analyze the performance of communication systems, see e.g. [7, 6].

[1] This is by far not a comprehensive list.

2 Channel Model

Consider the communication over a wireless channel with K transmit and N receive antennas and distortion by additive white Gaussian noise. Such a channel can be described in the complex baseband as

$$\boldsymbol{r}[\mu] = \sum_{\nu} \boldsymbol{H}[\nu]\boldsymbol{x}[\mu-\nu] + \boldsymbol{n}[\mu] \qquad (1)$$

where μ denotes discrete time, the $K \times 1$ vector sequence $\boldsymbol{x}[\cdot]$ and the $N \times 1$ vector sequence $\boldsymbol{r}[\cdot]$ contain the transmit and receive signals at the K and N antenna elements in its respective components, $\boldsymbol{H}[\cdot]$ is the matrix-valued impulse response of the channel, and $\boldsymbol{n}[\cdot]$ is additive, temporally and spatially white Gaussian noise with zero mean and variance σ_0^2. With appropriate signal processing, e.g. orthogonal frequency-division multiplexing, the dispersive channel can be transformed into a memoryless channel. Thus, $\boldsymbol{H}[\nu] = \boldsymbol{0}$, $\forall \nu \neq 0$ is assumed to simplify further considerations and the discrete time index is dropped.

The matrix $\boldsymbol{H}$ models the propagation from the transmit array via the scattering objects to the receive array. Thus, it can be decomposed into the product of at least three matrices

$$\boldsymbol{H} = \boldsymbol{\Phi}^\dagger \boldsymbol{A} \boldsymbol{\Theta} \qquad (2)$$

where the $S \times K$ matrix $\boldsymbol{\Theta}$ accounts for the propagation from the transmit array to the S scattering objects, the $S \times N$ matrix $\boldsymbol{\Phi}$ accounts for the propagation from the scattering objects to the receive array and the (diagonal dominant) $S \times S$ matrix $\boldsymbol{A}$ accounts for propagation between and attenuation at scattering objects.

For the sake of analytical tractability we make two additional assumptions: 1) The entries of the matrices $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$ are independent identically distributed Gaussian random variables with zero mean and variances $1/N$ and $1/S$, respectively. This is a good approximation for many practical scenarios as shown in [5]. 2) The sizes of the matrices grow large with their ratios

$$\beta \triangleq \frac{K}{N}, \qquad \rho \triangleq \frac{S}{N} \tag{3}$$

remaining fixed. This approximation has also been made in context of antenna arrays in [4, 5] and is well-established for analysis of code-division multiple-access systems, see e.g. [8].

3 Replica Analysis

Consider the singular value decomposition

$$\boldsymbol{\Phi}^\dagger \boldsymbol{A} = \boldsymbol{U}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{V}^\dagger \tag{4}$$

where $\boldsymbol{U}, \boldsymbol{V}$ are unitary. The positive semi-definite diagonal matrix $\boldsymbol{\Lambda}$ contains the N eigenvalues of the $N \times N$ matrix $\boldsymbol{\Phi}^\dagger \boldsymbol{A}\boldsymbol{A}^\dagger \boldsymbol{\Phi}$. Note that the signal

$$\tilde{\boldsymbol{r}} \triangleq \boldsymbol{U}^\dagger \boldsymbol{r} = \sqrt{\boldsymbol{\Lambda}}\boldsymbol{V}^\dagger \boldsymbol{H}\boldsymbol{x} + \boldsymbol{U}^\dagger \boldsymbol{n} \tag{5}$$

is a sufficient statistic for $\boldsymbol{r}$. Since the matrix $\boldsymbol{U}$ is unitary, $\tilde{\boldsymbol{n}} \triangleq \boldsymbol{U}^\dagger \boldsymbol{n}$ is independent identically distributed spatially and temporally white Gaussian noise. Moreover, since $\boldsymbol{V}$ is unitary as well and $\boldsymbol{\Theta}$ is composed of independent identically distributed Gaussian entries by assumption, the matrix $\tilde{\boldsymbol{\Theta}} \triangleq \boldsymbol{V}^\dagger \boldsymbol{\Theta}$ is composed of independent identically distributed Gaussian random entries as well. Note also that all involved signals live in a vector space with at most N dimensions. Thus, we may ignore potential null spaces in $\boldsymbol{\Lambda}$ and $\tilde{\boldsymbol{\Theta}}$ occuring if $S > N$ and arrive without loss of generality at the equivalent channel model

$$\tilde{\boldsymbol{r}}' = \sqrt{\boldsymbol{\Lambda}'}\tilde{\boldsymbol{\Theta}}'\boldsymbol{x} + \tilde{\boldsymbol{n}} \tag{6}$$

where $\cdot'$ denotes the removal of the potential null spaces such that $\tilde{\boldsymbol{r}}'$ and $\tilde{\boldsymbol{n}}'$ are $N \times 1$ vectors and $\boldsymbol{\Lambda}'$ is an $N \times N$ matrix. Note that it is not possible to apply a similar decomposition to the matrix $\tilde{\boldsymbol{\Theta}}'$ and introduce an equivalent transmit vector $\tilde{\boldsymbol{x}}$, since the transmit vector $\boldsymbol{x}$ is assumed to be binary. Note also that in order to ease calculations, we restrict ourselves to a real-valued channel in the following.

The replica method is a tool developed in statistical physics [6]. The key to study such systems in statistical physics is the *free energy* (normalized to the number of transmit antennas)

$$\mathcal{F}_K(\boldsymbol{r}, \boldsymbol{H}) = \frac{1}{K}\log f(\boldsymbol{r}, \boldsymbol{H}) + \frac{N \log c_0}{K} \tag{7}$$

where $f(\boldsymbol{r}, \boldsymbol{H})$ is the probability density function of the channel output signal and $c_0 \triangleq \sqrt{2\pi}\sigma_0$ is some normalization constant. A fundamental principle of statistical physics states that the free energy is *self-averaging* in the large system limit, i.e.

$$\lim_{K\to\infty} \frac{1}{K}\mathcal{F}_K(\boldsymbol{r}, \boldsymbol{H}) = \mathcal{F}. \tag{8}$$

This means that in the large system limit, the free energy becomes independent of the realizations of the random processes $\boldsymbol{r}$ and $\boldsymbol{H}$. Since the self-averaging property implies that the expectation of the free energy is identical to the free energy itself, i.e. $\mathsf{E}\,\mathcal{F} = \mathcal{F}$, the free energy is (up to the sign and some constant) equivalent to the differential entropy (in information-theoretic sense) of the channel output signal per antenna

$$\lim_{K\to\infty} \frac{1}{K} h(\boldsymbol{r}) = \frac{\log c}{\beta} - \mathcal{F}. \tag{9}$$

This relation will be helpful to find the channel capacity in Section 4

Using the replica method, it is shown in the Appendix that the free energy is given by

$$\begin{aligned}\mathcal{F} &= \int_{\mathrm{IR}} \log\left[\cosh\left(\sqrt{E}z + E\right)\right] Dz - \frac{E}{2}(1+m) \\ &\quad -\frac{1}{2\beta}\left[1 + \int \log[1 + \lambda\frac{\beta}{\sigma_0^2}(1-m)]\, dF_\lambda(\lambda)\right]\end{aligned} \tag{10}$$

with the macroscopic parameters m and E being defined by the following system of equations:

$$m = \int_{\mathrm{IR}} \tanh\left(\sqrt{E}z + E\right) Dz \tag{11}$$

$$E = \int \frac{\lambda}{\sigma_0^2 + \lambda\beta(1-m)}\, dF_\lambda(\lambda) \tag{12}$$

$Dz \triangleq e^{-z^2/2}dz/\sqrt{2\pi}$ and

$$F_\lambda(x) = \lim_{N\to\infty} \frac{1}{N}\left|\{\lambda \in \mathcal{L} : x < \lambda\}\right| \tag{13}$$

denoting the limiting distribution of the set $\mathcal{L}$ containing the eigenvalues of $\boldsymbol{\Phi}^\dagger \boldsymbol{A}\boldsymbol{A}^\dagger \boldsymbol{\Phi}$ (which are identical to the eigenvalues of $\boldsymbol{\Lambda}'$). It is shown in [9, 10] that $F_\lambda(x)$ is self-averaging as well and how it can be calculated for arbitrary matrices $\boldsymbol{A}$. For the particular case of $\boldsymbol{A} = \mathbf{I}$, the eigenvalue density can be given explicitly and reads [8, Proposition 2.1]

$$\begin{aligned}f_\lambda(x) &= \frac{d}{dx}F_\lambda(x) \\ &= \frac{\sqrt{4\rho - (\rho x - \rho - 1)^2}}{2\pi x} + \max\{1-\rho, 0\}\,\delta(x)\end{aligned} \tag{14}$$

for $(1-\sqrt{\rho})^2/\rho < x < (1+\sqrt{\rho})^2/\rho$ and 0 elsewhere. In that case, the integrals over the eigenvalue distributions in (10) and (12) can be evaluated explicitly using methods developed in [11] and we find

$$E = \frac{1}{\beta(1-m)}\left(\frac{\rho}{2s} + \frac{1+\rho}{2} - \sqrt{\frac{(1-\rho)^2}{4} + \frac{\rho(1+\rho)}{2s} + \frac{\rho^2}{4s^2}}\right) \tag{15}$$

$$\int \log[1+\lambda\frac{\beta}{\sigma_0^2}(1-m)]\,dF_\lambda(\lambda) =$$
$$\log(1+s-t)+\rho\log(1+s/\rho-t)-\rho\,\frac{s}{t} \quad (16)$$

with

$$s \triangleq \frac{\beta}{\sigma_0^2}(1-m) \quad (17)$$

and

$$t \triangleq \frac{1}{4\rho}\left(\sqrt{\rho+s(\sqrt{\rho}+1)^2}-\sqrt{\rho+s(\sqrt{\rho}-1)^2}\right)^2. \quad (18)$$

For most other choices of $\boldsymbol{A}$ such explicit formulas are not possible, but fixed point equations can be found which determine E.

If the system of equations (11) and (12) has multiple solutions, the correct solution is those one for which the free energy is maximum.

4 Channel Capacity

Channel capacity per transmit antenna for binary input alphabet is given in terms of differential entropies $h(\cdot)$ by

$$C=\frac{h(\boldsymbol{r})-h(\boldsymbol{r}|\boldsymbol{x})}{K}=\frac{h(\boldsymbol{r})-h(\boldsymbol{n})}{K} \quad (19)$$

where $h(\boldsymbol{n})=\frac{N}{2}(1+\log c)$. Thus, channel capacity relates in the large system limit to the free energy like [7]

$$\lim_{K\to\infty} C = -\mathcal{F}-\frac{1}{2\beta}. \quad (20)$$

Note that (20) holds for any choice of the constant c.

Channel capacity is plotted in Fig. 1 versus load β and richness ρ for $\boldsymbol{A}=\mathbf{I}$ making use of (15). Comparing Fig. 1 with its counterpart for Gaussian input alphabet [5, Fig. 5][2], we find only one significant difference: For small loads, the binary input capacity is degraded due to the limitation of the input entropy which cannot exceed 1 bit ($\log 2 \approx 0.69$ nats) per transmit antenna. This area is highlighted by bold grid in Fig. 1.

The limited input entropy of binary signaling does also affect the optimal partitioning of antennas into tranmit and receive antenna. While for Gaussian input signals, it is best (from a capacity point of view) to have equal number of transmit and receive antennas, for binary signals a different trade-off occurs as Fig. 2 shows. Particularly for rich scattering, it is beneficial to have more transmit antennas than receive antennas given a fixed total number of antennas at both ends altogether. This is, as entropy per transit antenna is upper bounded by 1 bit, while the enropy per receive antenna is only limited by the (logarithm of the) number of the quantization levels of the analog to digital converter, in practice, and unlimited in theory. (Of course, a significant part of the channel output entropy is noise entropy.)

[2] The capacity in [5] appears to be twice the capacity in Fig. 1, since in [5] a complex channel was addressed.

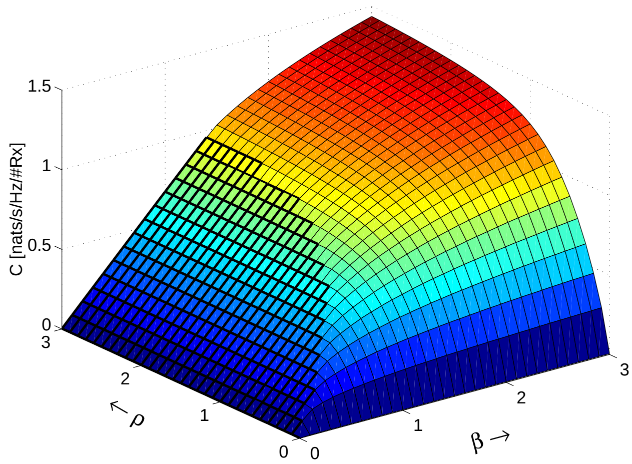

Figure 1: Channel capacity per receive antenna vs. load and richness for binary input alphabet for 9 dB SNR per transmit antenna.

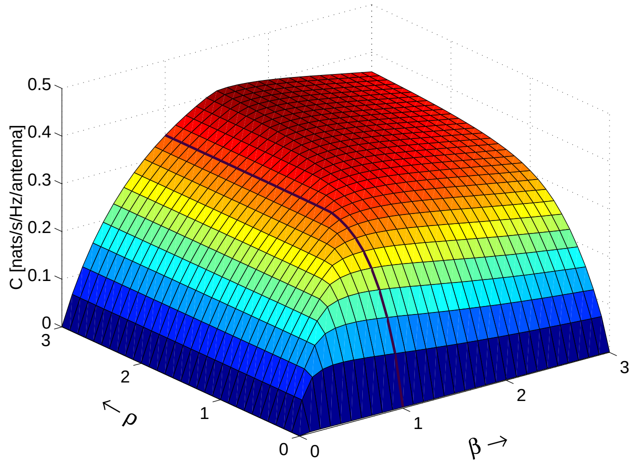

Figure 2: Channel capacity per total number of antennas vs. load and richness for binary input alphabet for 9 dB SNR per transmit antenna.

5 Bit Error Probability

The individually optimum detector, i.e. the one which minimizes bit error probability, performs an exhaustive search over all possibly transmitted vectors $\boldsymbol{x}$. It is given by [8]

$$\hat{\boldsymbol{x}} = \underset{\xi}{\operatorname{argmax}} \sum_{\boldsymbol{x}:x_k=\xi} \mathrm{e}^{-\frac{1}{2\sigma_0^2}||\boldsymbol{r}-\boldsymbol{H}\boldsymbol{x}||^2}. \quad (21)$$

The bit error probability can be shown to converge to

$$\int_{\sqrt{E}}^{\infty} Dz \quad (22)$$

in the large system limit [7].

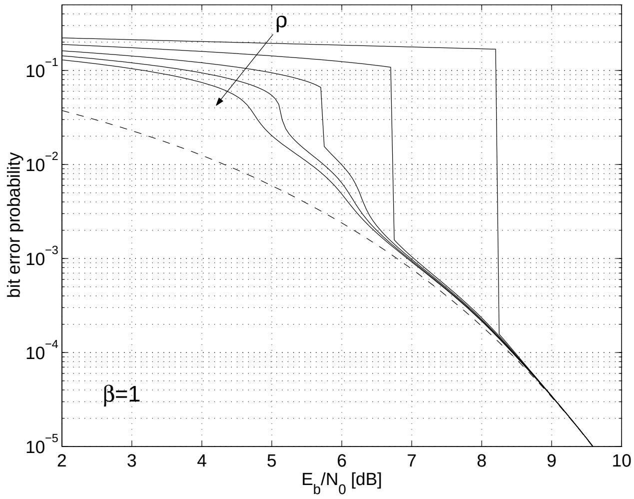

Figure 3: Bit error probability of individually optimum detection vs. signal-to-noise ratio for various richnesses $\rho = 0.5, 0.7, 1, 1.4, 2$ (following arrow) and load $\beta = 1$. The dashed line refers to an orthogonal channel (e.g. $\boldsymbol{H} = \mathbf{I}$).

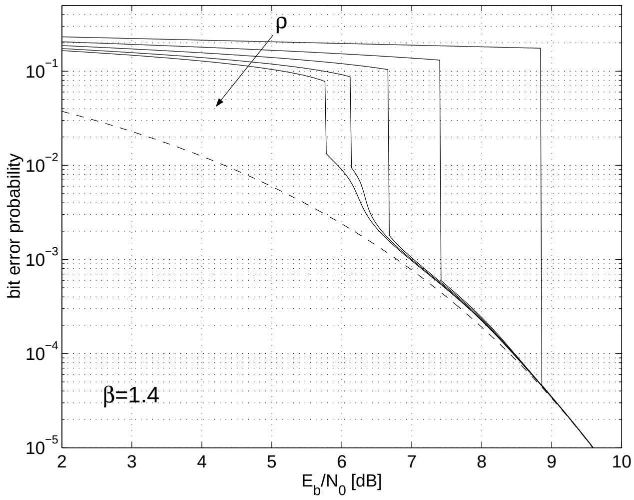

Figure 4: Bit error probability of individually optimum detection vs. signal-to-noise ratio for various richnesses $\rho = 0.7, 1, 1.4, 2, 2.8$ (following arrow) and load $\beta = 1.4$. The dashed line refers to an orthogonal channel (e.g. $\boldsymbol{H} = \mathbf{I}$).

Bit error probability is depicted for equal number of transmit and receive antennas in Fig. 3. It shows a *waterfall* behavior unless scattering is very rich. Such waterfall behavior was observed also in overloaded code-division multiple-access (CDMA) [7] and turbo coding [12]. Surprisingly, rich scattering ($\rho > 1$) is not important if low bit error rates are targeted. However, richer scattering shifts the waterfall region towards lower signal-to-noise ratios or makes it even disappear.

The importance of rich scattering increases if the system is overloaded ($\beta > 1$), cf. Fig. 4, since rich scattering helps to keep the waterfall region at a reasonable signal-to-noise ratio level and compensate to a certain extent for increasing load.

6 Conclusions

The individually optimum detector shows a surprising robustness against poor scattering for small and moderate loads which is due to the occurence of the waterfall behavior of the uncoded bit error probability.

In rich scattering, more transmit than receive antennas are needed in order to achieve high channel capacity with binary modulation.

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Appendix

The proof is a generalization of the derivation in [7]. Since the full proof is very lengthy, some intermediate steps whose generalizations in comparison to [7] are obvious are omitted in this appendix. For the convenience of the reader, the appendix uses the same notation as reference [7].

Let $f_\sigma(\boldsymbol{r}, \boldsymbol{H}, \sigma)$ be the pdf of the output of a virtual channel which is identical to the one considered except that the variance of the additive noise is σ^2 instead of σ_0^2. This enables us to re-write the free energy (7) as

$$\mathcal{F}_K = \lim_{\sigma \to \sigma_0} \lim_{n \to 0} \frac{\partial}{\partial n} \underbrace{\frac{1}{K} \log \int\limits_{\mathbb{R}^N} c^{nN} \overline{f_{\sigma_0}(\boldsymbol{r}, \boldsymbol{H}, \sigma_0) f_\sigma^n(\boldsymbol{r}, \boldsymbol{H}, \sigma)}\, d\boldsymbol{r}}_{\triangleq \Xi_n} \quad (23)$$

where the overline denotes averaging with respect to $\boldsymbol{H}$.

The essential trick of the replica method is to evaluate the integral Ξ_n for integer values of n only, though an expression for real n is required to perform the limit operation in (23). Though this step still lacks rigorous mathematical justification—it is actually a topic of intense research in mathematical physics—the replica method was found to produce reasonable and accurate results, particularly for analysis of channel capacities and bit error rates in [7]. For further discussion on this matter, see [13, 13].

Applying the replica trick, the integral is given by (24), see top of next page, and differs from [7, (20)] by $\lambda_c \neq 1$.

In order to perform the integration in (24), the n-dimensional space is split into subshells Q where the inner

$$\Xi_n = \sum_{\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n} \left(\prod_{a=0}^{n} p(\boldsymbol{x}_a)\right) \times \tag{24}$$
$$\prod_{c=1}^{N} \frac{\int_{\mathbb{R}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\tilde{r}' - \sqrt{\frac{\lambda_c}{N}}\sum_{k=1}^{K} s_k x_{0k}\right)^2\right] \prod_{a=1}^{n} \exp\left[-\frac{1}{2\sigma^2}\left(\tilde{r}' - \sqrt{\frac{\lambda_c}{N}}\sum_{k=1}^{K} s_k x_{ak}\right)^2\right] d\tilde{r}'}{\sqrt{2\pi}\sigma_0}$$

product of two different vectors $\boldsymbol{x}_a$ and $\boldsymbol{x}_b$ is constant. Thus, we find

$$\Xi_n = \int_{[-1,1]^{n(n+1)/2}} \exp\left[\sum_{c=1}^{N} \mathcal{G}_c\{Q\}\right] \mu_K\{Q\} \left(\prod_{a<b} dQ_{ab}\right), \tag{25}$$

a generalization of [7, (22)], where $\mu_K\{Q\}$ denotes the probability weight of the subshell and

$$e^{\mathcal{G}_c\{Q\}} = \tag{26}$$
$$\frac{1}{\sqrt{2\pi}\sigma_0}\int_{\mathbb{R}} \exp\left[-\frac{\beta}{2\sigma_0^2}\left(\tilde{r}'/\beta^{1/2} - \sqrt{\lambda_c}v_0\{Q\}\right)^2\right] \times \prod_{a=1}^{n} \exp\left[-\frac{\beta}{2\sigma^2}(\tilde{r}'/\beta^{1/2} - \sqrt{\lambda_c}v_a\{Q\})^2\right] d\tilde{r}'$$
$$= +\mathcal{O}(K^{-1}) \tag{27}$$

with the correlated jointly Gaussian random variables $v_a \triangleq K^{-\frac{1}{2}}\sum_k s_k x_{ak}, a \in \{0,1,\ldots,n\}$.

In the limit of $K \to \infty$ one of the exponential terms in (25) will dominate over all others. As shown in [7], the replicas within the dominant subshell are symmetric. Thus, we can assume without loss of generality $Q_{0a} = m, \forall a \neq 0$, and $Q_{ab} = q, \forall 0 \neq a \neq b \neq 0$. This allows to construct the correlated Gaussian random variables v_a out of iindependent zero-mean, unit-variance Gaussian random variables u, t, z_a. With that substitution and the definitions $B \triangleq \beta/\sigma^2, B_0 \triangleq \beta/\sigma_0^2$, we get (29), the analogy to [7, (28) and (29)]. Obviously, simply an additional factor λ_c appears in front of B and B_0.

Under the replica symmetry (RS) assumption, we solve the integral in (25) (generalizing [7, (37)])

$$\lim_{K\to\infty} K^{-1}\log\Xi_n = \sup_{m,q}\left\{\lim_{N\to\infty}\beta^{-1}\frac{1}{N}\sum_{c=1}^{N}\mathcal{G}_c(m,q) - \mathcal{I}(m,q)\right\} \tag{30}$$

using the saddle-point method with the rate function $\mathcal{I}(m,q)$.

A supremum point with respect to m and q satisfies the extremum condition derived from (29) and [7, (34)], which is given by (31) and (32). With the asymptotic convergence of the eigenvalue distribution

$$\lim_{N\to\infty}\frac{1}{N}\sum_{c=1}^{N} f(\lambda_c) = \int f(\lambda)dF_\lambda(\lambda), \tag{33}$$

we proceed the same way as [7] and finally find

$$\mathcal{F}^{\mathrm{RS}} = \int_{\mathbb{R}} \log[\cosh(\sqrt{F}z + E)]Dz - Em - \frac{1}{2}F(1-q)$$
$$-\frac{1}{2\beta}\int\left[\log[1+\lambda B(1-q)] \right. \tag{34}$$
$$\left. + \frac{B/B_0 + \lambda B(1-2m+q)}{1+\lambda B(1-q)}\right] dF_\lambda(\lambda)$$

with the macroscopic parameters m, q, E, F being defined by the following system of equations:

$$m = \int_{\mathbb{R}} \tanh\left(\sqrt{F}z + E\right) Dz$$
$$q = \int_{\mathbb{R}} \tanh\left(\sqrt{F}z + E\right) Dz$$
$$E = \beta^{-1}\int \frac{\lambda B}{1+\lambda B(1-q)} dF_\lambda(\lambda)$$
$$F = \beta^{-1}\int \frac{\lambda B^2/B_0 + \lambda^2 B^2(1-2m+q)}{[1+\lambda B(1-q)]^2} dF_\lambda(\lambda) \tag{35}$$

Letting $\sigma \to \sigma_0$ which implies $B \to B_0 = \beta/\sigma^2, q \to m, F \to E$, we find the desired result.

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$$e^{\mathcal{G}_c(m,q)} = \frac{1}{\sqrt{2\pi}\sigma_0} \int\limits_{\mathrm{IR}^2} \int\limits_{\mathrm{IR}} \exp\left[-\frac{B_0}{2}\left(u\sqrt{\lambda_c(1-m^2/q)} - tm\sqrt{\frac{\lambda_c}{q}} - \tilde{r}'/\beta^{1/2}\right)^2\right] Du \tag{28}$$

$$\times \left[\int\limits_{\mathrm{IR}} \exp\left[-\frac{B}{2}\left(z\sqrt{\lambda_c(1-q)} - t\sqrt{\lambda_c q} - \tilde{r}'/\beta^{1/2}\right)^2\right] Dz\right]^n Dt\, d\tilde{r}'$$

$$= \sqrt{\frac{(1+\lambda_c B(1-q))^{(1-n)}}{1+\lambda_c B(1-q) + n\lambda_c B(\lambda_c^{-1}B_0^{-1} + 1 - 2m + q)}} \tag{29}$$

$$E = \lim_{N\to\infty} \frac{1}{N} \sum_{c=1}^{N} \frac{\beta^{-1}\lambda_c B}{1+\lambda_c B(1-q) + n\lambda_c B(\lambda_c^{-1}B_0^{-1} + 1 - 2m + q)} \tag{31}$$

$$F = \lim_{N\to\infty} \frac{1}{N} \sum_{c=1}^{N} \frac{\beta^{-1}\lambda_c^2 B^2(\lambda_c^{-1}B_0^{-1} + 1 - 2m + q)}{\{1+\lambda_c(1-q) + n\lambda_c B[\lambda_c^{-1}B_0^{-1} + 1 - 2m + q]\}[1+\lambda_c B(1-q)]} \tag{32}$$

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