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Vector Precoding for Singular MIMO Channels

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Abstract

We propose a method for vector precoding suited to singular multiple-input multiple-output (MIMO) channels were the number of data streams is larger than the number of transmit antennas. We calculate the probability that our method, due to the channel singularity, fails to provide the data without crosstalk to the receiver, analytically, and show that this probability goes to zero exponentially with the size of the MIMO system if the number of QPSK data streams in smaller than twice the number of transmit antennas.

1. INTRODUCTION

Vector precoding for wireless MIMO systems has been an active area of research over the past years [1, 2, 3, 4]. It is driven by low cost applications were the receiver cannot afford sophisticated spatial signal processing due to complexity constraints. Thus, interference mitigation is to be performed at the transmitter site.

Many precoding schemes consist of two steps. At the transmitter the channel matrix is inverted and the transmitted signal is pre-distorted with the (pseudo-)inverse of the MIMO channel matrix such that there will not be any crosstalk at the receiver side and different data streams can be processed independently of each other. Since such a pre-distortion amplifies the power of the transmitted signal without providing additional separation of signals at the receiver, the signal is often pre-processed in a nonlinear fashion before sent into the linear pre-distortion. The non-linear preprocessing serves the purpose to reduce the power amplification caused by the linear predistortion. This is achieved by forming signal vectors that, with high probability, lie in subspaces of the predistortion matrix with small eigenvalues.

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At first sight, one might be tempted to conclude that such a precoding procedure, due to the inversion of the channel, requires a channel matrix with rank K or greater in order to precode for K data streams and receive them free of crosstalk at the receiver. However, there is a way around that requirement: If the nonlinear preprocessing is able to shape all of the transmitted vectors in such a way that they do not intersect the null space of the channel, the fact that the channel is singular or rank-deficient does not affect the ability to pre-distort those vectors in such a way that they are received without crosstalk.

In a recent work [4], it was shown that nonlinear processing can indeed shape the signal vectors in such a favorable way if the number of receive and transmit antennas grows to infinity with a given ratio α and this ratio is not too large. However, in many cases the critical ratio was found to be larger than 1. Thus allowing for more data streams than the rank of the channel.

In this work, we show that convex precoding, proposed and analyzed in the large system limit in [4], can fulfill the promise of working with singular channels even for finite rank deficient channel matrices with high probability. Admittedly, we cannot guarantee to find a suitable vector outside the null space of the channel for an realization of the channel matrix. If such an outage happens, however, with sufficiently low probability, e.g. with lower probability than other deleterious channel impairments, it does not pose a problem in practice.

2. SYSTEM MODEL

Consider a MIMO system described by

$$\boldsymbol{r} = \boldsymbol{H}\boldsymbol{t} + \boldsymbol{n} \tag{1}$$

where the vectors $\boldsymbol{t}$ and $\boldsymbol{r}$ denote the signals at the multiple transmit and receive antennas, respectively, $\boldsymbol{H}$ denotes the $K \times N$ matrix of antenna gains, and $\boldsymbol{n}$ denotes white Gaussian noise. The transmit vector $\boldsymbol{t}$

is formed by a linear transformation with the pseudo-inverse of the channel

$$\boldsymbol{t} = \boldsymbol{H}^{+}\boldsymbol{x} = \lim_{\epsilon\to 0} \boldsymbol{H}^{\dagger}\left(\boldsymbol{H}\boldsymbol{H}^{\dagger} + \epsilon\mathbf{I}\right)^{-1}\boldsymbol{x} \qquad (2)$$

out of the precoded vector $\boldsymbol{x}$. The precoded vector $\boldsymbol{x}$ is chosen to uniquely represent the data vector $\boldsymbol{s}$.

3. CONVEX PRECODING

In this work, the data vector will be quaternary $\boldsymbol{s} \in \{\pm 1 \pm \mathrm{j}\}^{\mathrm{K}}$ and the precoded vector $\boldsymbol{x} = [x_1, x_2, \ldots, x_K]$ fulfills

$$\Re(x_k)\Re(s_k) \geq 1 \leq \Im(x_k)\Im(s_k) \qquad \forall k. \qquad (3)$$

This allows to find the precoded vector by convex quadratic programming. Let $\mathcal{X}(\boldsymbol{s})$ denote the set of all vectors $\boldsymbol{x}$ such that (3) holds. Then, the precoded vector that minimizes the transmitted power for a given data vector $\boldsymbol{s}$ is given by

$$\boldsymbol{x} = \underset{\boldsymbol{\xi}\in\mathcal{X}(\boldsymbol{s})}{\operatorname{argmin}} \lim_{\epsilon\to 0} \boldsymbol{\xi}^{\dagger}\left(\boldsymbol{H}\boldsymbol{H}^{\dagger} + \epsilon\mathbf{I}\right)^{-1}\boldsymbol{\xi}. \qquad (4)$$

If the limit $\epsilon \to 0$ in (4) exists, the precoded vector $\boldsymbol{x}$ lies in the space spanned by the channel, i.e.

$$\boldsymbol{x} \in \operatorname{Span}\left(\boldsymbol{H}\boldsymbol{H}^{\dagger}\right), \qquad (5)$$

and the received signal is free of interference and solely disturbed by additive noise

$$\boldsymbol{r} = \boldsymbol{x} + \boldsymbol{n}. \qquad (6)$$

The limit in (4) clearly always exists, if the matrix $\boldsymbol{H}\boldsymbol{H}^{\dagger}$ has full rank. If $\boldsymbol{H}$ is a random matrix with independent Gaussian entries, it has full rank with probability 1, if and only if $K \leq N$.

4. SINGULAR CHANNELS

Let us now focus on the question whether the limit $\epsilon \to 0$ exists for singular channel matrices, i.e.

$$\operatorname{rank}\left(\boldsymbol{H}\boldsymbol{H}^{\dagger}\right) < K. \qquad (7)$$

Hereby, we restrict the considerations to channel matrices $\boldsymbol{H}$ with entries that are independent complex Gaussian random variables with zero mean and variance $1/N$.

This question was answered in the large matrix limit $K = \alpha N \to \infty$ with fixed ratio α in [4] where the transmitted energy per symbol E was shown to be the solution to

$$\mathrm{Q}\left(\sqrt{\frac{2}{\alpha E}}\right) = \frac{2 + (\alpha - 1)E + \sqrt{\frac{\alpha E}{\pi}}\,\mathrm{e}^{-\frac{1}{\alpha E}}}{2 + \alpha E} \qquad (8)$$

where $\mathrm{Q}(x) = \int_x^\infty \exp(-\xi^2/2)\mathrm{d}\xi/\sqrt{2\pi}$. Since the energy per symbol diverges to infinity at the threshold ratio α, the Q-function can be developed into a series for small argument, and it can be shown that the energy stays finite as long as $\alpha < 2$. Thus, in the limit of a very large number of antennas, the channel need not have full rank, but only half rank.

We now answer the question for matrices of finite size. The question is equivalent to calculating the probability that there is no interference

$$\begin{aligned} P(\boldsymbol{s}) &= \Pr(\boldsymbol{r} = \boldsymbol{x} + \boldsymbol{n}|\boldsymbol{s}) & (9)\\ &= \Pr\left(\mathcal{X}(\boldsymbol{s}) \cap \operatorname{Span}(\boldsymbol{H}\boldsymbol{H}^{\dagger}) \neq \emptyset\right) & (10)\end{aligned}$$

Note that by the unitary invariance of $\boldsymbol{H}\boldsymbol{H}^{\dagger}$, the probability does not depend on the data vector $\boldsymbol{s}$ and we choose without loss of generality $s_k = 1 + \mathrm{j}, \forall k$, i.e. $\boldsymbol{s} = \mathbf{1} + \mathrm{j}\mathbf{1}$. Moreover, the span is invariant to the norm of its bases. Thus, it is irrelevant whether

$$\Re(x_k) \geq 1 \leq \Im(x_k), \qquad \forall k \qquad (11)$$

or just

$$\Re(x_k) > 0 < \Im(x_k), \qquad \forall k. \qquad (12)$$

In this respect the following result from [5] turns out helpful:

Theorem 1 *Let $\boldsymbol{R}$ be an $N \times K$ matrix composed of independent identically distributed real Gaussian random variables, then the probability that* $\operatorname{span}(\boldsymbol{R})$ *contains a vector whose components are all positive is given by*

$$2^{1-K} \sum_{\ell=0}^{N-1} \binom{K-1}{\ell}. \qquad (13)$$

The result can be extended to the complex case [6]:

Corollary 1 *Let $\boldsymbol{C}$ be an $N \times K$ matrix composed of independent identically distributed complex Gaussian random variables, then the probability that* $\operatorname{span}(\boldsymbol{C})$ *contains a vector for which both real parts and imaginary parts of all components are positive is given by*

$$P(K, N) = 2^{1-2K} \sum_{\ell=0}^{2N-1} \binom{2K-1}{\ell}. \qquad (14)$$

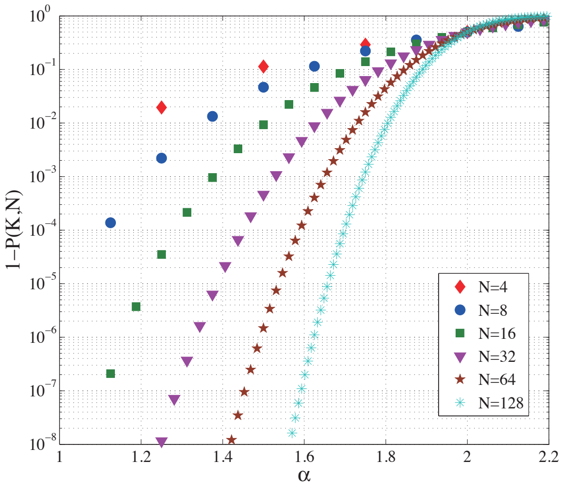

Figure 1: Probability that convex precoding fails to find a solution vs. the ratio of number of receive to number of transmit antennas.

Note that for matrices with half rank, i.e. $K = 2N$, we find

$$P(2N, N) = \frac{1}{2} \tag{15}$$

and

$$P(K, K-N) = 1 - P(K, N). \tag{16}$$

The probability that convex precoding fails to find a solution is plotted in Fig. 1. It can be observed that a significant overload is tolerable while keeping the failure probability small. This holds particularly for large number of antennas. In systems with subsequent error control coding, where failure probabilities of even a few percent are tolerable, a MIMO system could be at 50% to 80% overloaded with convex precoding.

It is worth to be remarked that a failure does not mean that all data bits are lost. It just means that it is impossible to send all data bits of a symbol vector $\boldsymbol{s}$ free of interference. It is still possible and actually quite likely that some of the data bits can be successfully be precoded for. To find the probability of failure on the basis of the components of the data vector $\boldsymbol{s}$ is a harder problem which is left for future research.

5. CONCLUSIONS

We have presented a method for vector precoding on singular MIMO channels and shown that the method can be used to increase the number of transmitted data streams significantly beyond the number of transmit antennas. The extend to which that is possible depends on the number of antennas in the system and the tolerable outage probability.

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