Minimum Bit Error Probability of Large Randomly Spread MC-CDMA Systems in Multipath Rayleigh Fading

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***Abstract*— We address bit error probability of the optimum multiuser detector for multi-carrier code-division multiple-access in frequency-selective fading channels. We use the replica method, a tool from statistical physics, to calculate the bit error probability in the large system limit for randomly assigned spreading sequences. The analysis allows for binary input symbols with biased probabilities and, therefore, can be used to analyze iterative multiuser decoders.**

I. Introduction

[1] In a recent landmark paper, Tanaka [1] presented an approximate analysis of randomly-spread direct sequence (DS) code-division multiple access (CDMA) with optimum detection with is accurate if the number of users and the spreading factor are large. His calculations were based upon the replica method developed in statistical physics [2], [3].

Due to the exponential complexity of the optimum multiuser detector, simulations are impossible for all systems with more than just a few users. Therefore, it is particularly important to have such an analysis at hand, if we want to compare state-of-the-art solutions against the best possible performance.

Tanaka's result was limited to the additive white Gaussian noise channel, but generalized by Guo and Verdú [4] to users with arbitrary powers and, therefore, implicitly to flat fading channels. In order to cope with the feedback information in iterative multiuser decoding, Müller et al. [5] generalized the result of Guo and Verdú to biased data symbols. In this paper, the previous results are generalized to multi-carrier (MC) CDMA in multipath fading channels. Examples of other recent applications of the replica method to the analysis of CDMA are [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

II. System Model

Consider a MC-CDMA system with K users and spreading factor N. Let $\boldsymbol{F}$ denote the $N \times N$ discrete Fourier transform (DFT) matrix, and $\boldsymbol{s}_k$ denote the spreading sequence of user k written as an $N \times 1$ vector. The system is modelled in matrix notation by

$$\tilde{\boldsymbol{r}} = \sum_{k=1}^{K} \boldsymbol{G}_k \boldsymbol{F} \boldsymbol{s}_k x_k + \tilde{\boldsymbol{n}} \tag{1}$$

where $\tilde{\boldsymbol{n}}$ denotes additive white Gaussian noise of variance σ_0^2, x_k denotes the data symbol of user k, $\tilde{\boldsymbol{r}}$ denotes the received signal, and the $N \times N$ circulant matrix $\boldsymbol{G}_k$ describes the multipath propagation channel of user k. In reality the convolution with the channel impulse response is linear, not cyclic like in (1). Since it is well-known that this fact can be easily accouted for by the introduction of a cyclic prefix, we will neglect it in the following to ease notation and improve readability of the paper. The extension of the results presented in this paper to MC-CDMA with real-world linear convolution and cyclic prefix is straightforward.

At the receiver, we first transform the signal into discrete frequency domain

$$\boldsymbol{r} = \boldsymbol{F}^{\mathrm{H}} \tilde{\boldsymbol{r}} \tag{2}$$

$$= \sum_{k=1}^{K} \boldsymbol{F}^{\mathrm{H}} \boldsymbol{G}_k \boldsymbol{F} \boldsymbol{s}_k x_k + \boldsymbol{n}. \tag{3}$$

The eigenvectors of any circulant matrix form a Fourier basis [18]. Thus, $\boldsymbol{F}^{\mathrm{H}} \boldsymbol{G}_k \boldsymbol{F}$ is diagonal, and (2) can be written as

$$\boldsymbol{r} = \underbrace{(\boldsymbol{W} \odot \boldsymbol{S})}_{\triangleq \boldsymbol{H}} \boldsymbol{x} + \boldsymbol{n} \tag{4}$$

where $\odot$ denotes elementwise multiplication, $\boldsymbol{S}$ is the $N \times K$ matrix of signature sequences, $\boldsymbol{x}$ is the vector of data symbols, and the k^{th} column of $\boldsymbol{W}$ is the channel transfer function of user k, i.e. the main diagonal of $\boldsymbol{F}^{\mathrm{H}} \boldsymbol{G}_k \boldsymbol{F}$.

We assume perfect channel estimation, i.e. perfect knowledge of the matrix $\boldsymbol{W}$. Thus, we can form the sufficient statistics

$$\boldsymbol{y} = \boldsymbol{H}^{\mathrm{H}} \boldsymbol{H} \boldsymbol{x} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{n}. \tag{5}$$

For binary transmission, an optimal multiuser detector will now detect for that vector $\hat{\boldsymbol{x}} \in \{+1; -1\}^K$ such that $\mathsf{E}\, \hat{\boldsymbol{x}}^{\mathrm{H}} \boldsymbol{x}$

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is maximum. Note that this optimality criterion is equivalent to minimizing the bit error probability.

In order to allow for large system analysis, we assume that all users use random spreading sequences with independent identically distributed random chips. Moreover, we assume that the distribution of the chips has variance $1/N$ and all its odd moments vanish. This implies that the entries of the matrix $\boldsymbol{H}$ are independent, but *not* identically distributed.

Following [14], we address binary transmission over a real-valued channel and assume that it performs asymptotically (in the number of users) identical to Gray-mapped quarternary phase-shift keying on the corresponding complex channel. This assumption has recently been proven true for flat fading channels in [17].

III. Free Energy

In order to calculate the minimum probability of error in the large system limit, we make use of an analogy of maximum likelihood detection and statistical mechanics. In particular, we use the *replica method* introduced to CDMA literature by Tanaka [1]. The replica method is a tool developed in statistical physics [2], [3]. The key to study such systems in statistical physics is the *free energy* (here normalized to the number of users)

$$\mathcal{F}_K(\boldsymbol{r},\boldsymbol{H}) = -\frac{1}{K}\log f(\boldsymbol{r},\boldsymbol{H}) - \frac{N\log u_0}{K} \tag{6}$$

where $f(\boldsymbol{r},\boldsymbol{H})$ is the probability density function of the channel output signal and $u_0 \triangleq \sqrt{2\pi}\sigma_0$ is some normalization constant. A fundamental principle of statistical physics states that the free energy is *self-averaging* in the large system limit, i.e.

$$\lim_{K\to\infty}\frac{1}{K}\mathcal{F}_K(\boldsymbol{r},\boldsymbol{H}) = \mathcal{F}. \tag{7}$$

This means that in the large system limit, the free energy becomes independent of the realizations of the random processes $\boldsymbol{r}$ and $\boldsymbol{H}$.

Using the replica method, it can be shown [19][2] that the free energy for binary data symbols with distribution

$$\mathrm{p}_{x_k}(x_k) = \frac{1+t_k}{2}\,\delta(x_k-1) + \frac{1-t_k}{2}\,\delta(x_k+1), \quad t_k\in[-1;1], \tag{8}$$

is given by [3]

$$\begin{aligned}\frac{\mathrm{F}(\boldsymbol{x})}{K} &= \frac{1}{2K}\sum_{c=1}^{N}\left[\sigma_0^2E_c - \log\left(\sigma_0^2E_c\right)\right]\\ &\quad -\frac{1}{K}\sum_{k=1}^{K}\frac{1}{2}\log\left(1-t_k^2\right) - \tilde{E}_k\\ &\quad +\int \tfrac{1+t_k}{2}\log\cosh\left(z\sqrt{\tilde{E}_k}+\tilde{E}_k+\tfrac{\lambda_k}{2}\right)+\\ &\quad \tfrac{1-t_k}{2}\log\cosh\left(z\sqrt{\tilde{E}_k}+\tilde{E}_k-\tfrac{\lambda_k}{2}\right)\frac{\mathrm{e}^{-\frac{z^2}{2}}\mathrm{d}z}{\sqrt{2\pi}}\end{aligned} \tag{9}$$

[2]The derivation in [19] was first given in a draft version of this paper, not in [19]. Due to page limitations it had to be omitted in this final version.

[3]The formulas in red are wrong in the Proc. of ISSSTA 2004 and have been corrected here.

with $\lambda_k = \tanh(t_k/2)$ and the macroscopic parameters $\tilde{E}_k$ and E_c being defined by the following fixed-point system of equations

$$\tilde{E}_k = \frac{1}{N}\sum_{c=1}^{N}E_c|w_{ck}|^2 \tag{10}$$

$$\begin{aligned}\frac{1}{E_c} &= \sigma_0^2 + \frac{\beta}{K}\sum_{k=1}^{K}\left(1-t_k^2\right)\frac{|w_{ck}|^2}{\sqrt{2\pi}}\\ &\quad \int\frac{1-\tanh\left(z\sqrt{\tilde{E}_k}+\tilde{E}_k\right)}{1-t_k^2\tanh^2\left(z\sqrt{\tilde{E}_k}+\tilde{E}_k\right)}\mathrm{e}^{-\frac{z^2}{2}}\mathrm{d}z.\end{aligned} \tag{11}$$

The system of equations (10) and (11) has, in general, more than one fixed point. Multiple fixed-points typically occur for a medium range of signal-to-noise ratios at high system loads β. The meaningful fixed-point with respect to the performance of the optimal multiuser detection (performing exhaustive search) is that one for which the free energy (9) is minimum. The meaningful fixed-point is either the smallest one, which is found if the fixed-point iteration is started with $\tilde{E}_k = 0, \forall k$, or the largest one, which is found if the fixed-point iteration is started with $\tilde{E}_k \to \infty, \forall k$.

The smallest fixed-point, even if the free energy is minimized by another fixed-point, has always an additional meaning not related to detection based on exhaustive search: It characterizes the performance that can be achieved with a nonlinear gradient-based search algorithm, also called a Hopfield neural network [20], [21]. Hopfield networks were proposed for detection of CDMA in [22], [23]. Surprisingly, they can approach the performance of the optimal[4] detector based on exhaustive search in some circumstances, i.e. if the free energy at the small fixed-point is smaller than the free energy at the large fixed-point. The observation that the performance of Hopfield networks significantly improves for large systems was already made by simulative means in [24], [25].

IV. Bit Error Probability

Following the considerations in [1], $\tilde{E}_k$ can be shown to be a kind of signal-to-interference and noise ratio, in the sense that the conditional bit error probability[5] of user k is given by

$$\Pr(\hat{x}_k \neq x_k|\boldsymbol{W}) = \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\tilde{E}_k}}^{\infty}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z. \tag{12}$$

In fact, it can even be shown that in the large system limit, an equivalent additive white Gaussian noise channel can be

[4]In neural networks literature, theterm *"optimal"* detector is often used for its neural network based approximation while, in multiuser detection literature and in this paper, the optimal detector refers to the exhaustive search algorithm.

[5]It is assumed that the bits of user k are equally likely $+1$ and -1, but the bits of the interfering users obey the distribution (8).

defined to model the multiuser interference [13]. The unconditional bit error probability is then simply given by

$$\Pr(\hat{x}_k \neq x_k) = \frac{1}{\sqrt{2\pi}} \mathop{\mathsf{E}}_{\boldsymbol{W}} \int_{\sqrt{\tilde{E}_k}}^{\infty} \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z. \tag{13}$$

The expectation over $\boldsymbol{W}$ could not be evaluated analytically, but was done by means of Monte-Carlo integration.

V. Special Cases

The parameter $\tilde{E}_k$ in (10) and (11) has the meaning of a signal-to-interference and noise ratio of user k. The meaning of the parameter E_c is somewhat more abstract. It is a performance measure for the channel at discrete frequency c which is somehow averaged over the user population.

A. Asymptotic Frequency-Invariance

There are practical cases where the frequency-dependency of E_c vanishes in the large-system limit. That is, (10) and (11) reduce to

$$\tilde{E}_k = P_k \left[\sigma_0^2 + \frac{\beta}{K}\sum_{n=1}^{K}\left(1-t_n^2\right)\frac{P_n}{\sqrt{2\pi}} \int \frac{1-\tanh\left(z\sqrt{\tilde{E}_n}+\tilde{E}_n\right)}{1-t_n^2\tanh^2\left(z\sqrt{\tilde{E}_n}+\tilde{E}_n\right)}\mathrm{e}^{-\frac{z^2}{2}}\mathrm{d}z\right]^{-1} \tag{14}$$

where the random variable P_k is given by

$$P_k = \frac{1}{N}\mathrm{trace}\left(\boldsymbol{G}_k^{\mathrm{H}}\boldsymbol{G}_k\right). \tag{15}$$

Note that (14) was already found in [5] and describes a flat fading channel with diversity that is equivalent to the multipath fading channel in (10) and (11).

Asymptotic frequency-invariance implies that the multiuser efficiency[6]

$$\eta_k = \frac{\tilde{E}_k\sigma_0^2}{P_k} \tag{16}$$

converges to a non-random deterministic constant, as the system size grows large which does not depend on the fading realization, but only on the fading statistics. Thus, the unconditional bit error probability can be given in fully analytical form without the need for Monte-Carlo integration

$$\Pr(\hat{x}_k \neq x_k) = \frac{1}{\sqrt{2\pi}\sigma_0}\int\int_{\sqrt{\eta_k P_k}}^{\infty}\exp\left(-\frac{z^2}{2\sigma_0^2}\right)\mathrm{d}z\mathrm{dF}(P_k). \tag{17}$$

The distribution of P_k can be calculated from (15).

Asymptotic frequency-invariance is not a pathological case. Consider, for instance, the uplink of a MC-CDMA system where the fading of all users is independent, but not necessary identically distributed. Then, any summation over the user population is independent of frequency, as the number of users approaches infinity. This holds, in particular, for the sum in (11). Independent fading for all users is not even a necessary condition for asymptotic frequency-invariance. Weaker conditions are given in [27, Chapter 4]. On the other hand, the downlink of a MC-CDMA system is frequency-variant as all users see the same channel and the user average is different from the ensemble average even as the number of users becomes large.

[6] See [26] for definition.

B. Fading with Product Statistics

If the fading matrix has unit rank,

$$\mathrm{rank}(\boldsymbol{W}) = 1, \tag{18}$$

the fading can be separated into a user dependent part and a frequency-dependent part

$$\boldsymbol{W} = \boldsymbol{f}\boldsymbol{u}^{\mathrm{T}} \quad \text{with} \quad ||\boldsymbol{f}||^2 = N. \tag{19}$$

Thus, we have

$$w_{ck} = u_k f_c. \tag{20}$$

This allows for the following simplification of (10) and (11):

$$\tilde{E}_k = \frac{|u_k|^2}{N}\sum_{c=1}^{N} E_c|f_c|^2 \tag{21}$$

$$= \frac{|u_k|^2}{N}\sum_{c=1}^{N}\left[\frac{\sigma_0^2}{|f_c|^2} + \frac{\beta}{K}\sum_{n=1}^{K}\left(1-t_n^2\right)\frac{|u_n|^2}{\sqrt{2\pi}} \int \frac{1-\tanh\left(z\sqrt{\tilde{E}_n}+\tilde{E}_n\right)}{1-t_n^2\tanh^2\left(z\sqrt{\tilde{E}_n}+\tilde{E}_n\right)}\mathrm{e}^{-\frac{z^2}{2}}\mathrm{d}z\right]^{-1} \tag{22}$$

This case is typical for the downlink of a MC-CDMA system where all users experience the same fading channel with discrete frequency response f_c and where the power control policy is frequency-independent and given by $\mathsf{E}\,|u_k|^2$.

Comparing (14) with (21), it becomes clear that, in frequency-selective fading, the downlink of randomly spread MC-CDMA shows worse performance than the uplink. In the downlink, deep fades in the spectrum make the term $\sigma_0^2/|f_c|^2$ in (21) become large at certain frequencies. This is like reducing the spreading factor, or equivalently increasing the load. In the uplik, however, frequency-dependency asymptotically vanishes in the equations for bit error rate, since parts of the spectrum that deeply fade for one user are easily utilized by some other users whose fading gaps appear at other frequencies. If the number of users approaches infinity, the all frequencies are utilized equally, not by the users individually, but by the total user population seen as a whole.

Generalizing (16), we can define a frequency-dependent multiuser efficiency for fading with product statistics as

$$\breve{\eta}_c = E_c\sigma_0^2|f_c|^2. \tag{23}$$

The multiuser efficiency, then, is given by

$$\eta = \frac{1}{N}\sum_{c=1}^{N}\breve{\eta}_c, \tag{24}$$

and is unique for all users.

1) Wideband case: If the number of independent propagation paths scales with the spreading factor, referred to as the wideband case, the multiuser efficiency becomes deterministic in the large system limit, and the bit error probability can be conveniently calculated via (17) with $P_k = |u_k|^2$.

The frequency-dependent multiuser efficiency $\breve{\eta}_c$ does not become deterministic in the large system limit, in general. However, the only randomness it contains in the large system limit, is the channel transfer function f_c, since all other random variables $\tilde{E}_n, t_n, u_n$ are averaged out as $K \to \infty$. We can even eliminate the dependence on f_c, if we define a sorting permutation $\pi(c)$ such that $\breve{\eta}_{\pi(1)} \leq \breve{\eta}_{\pi(2)} \leq \ldots \leq \breve{\eta}_{\pi(N)}$. Then, $\breve{\eta}_{\pi(c)}$ is (up to a normalizing scalar factor) the inverse cumulative distribution function of $\breve{\eta}_c$, as $N \to \infty$ and, thus, a deterministic, non-random quanitity in the large system limit. For calculation of the wideband multiuser efficiency with (24), the permutation is as good as the original frequency-dependent multiuser efficiency, since the terms of a sum commute.

Note that an equivalent permutation could also be applied in the general case of fading without product statistics in order to make the random variables $E_{\pi_k}(c)$ deterministic in the large system limit via a user dependent permutation $\pi_k(c)$. In order to avoid the Monte-Carlo integration in (13), we need the distribution of $|w_{ck}|^2$. Due to the central limit theorem, it is clear that $|w_{ck}|^2$ is exponentially distributed over c.

2) Narrowband case: If the number of propagation paths remains finite while the other system parameters, N, K grow above all bounds, the multiuser efficiency remains a random variable, since there are only a finite number of random variables determining the distribution f_c.

VI. Numerical Example

In order to give a numerical example, we chose load $\beta = \frac{K}{N} = 1.5$, unbiased priors, and users with identical signal-to-noise ratios (SNRs). We assumed a fading channel with L propagation paths. The L paths were independent Rayleigh fading taps with equal average power. Their delay times were 0 to $L-1$ chip intervals respectively. Fig. 1 shows that for SNRs larger than 5 dB, the bit error probability approaches single-user performance quickly. The threshold SNR seems to be independent of the number of propagation paths. Below the threshold SNR, diversity reception is useless.

VII. Remark

Assuming a Gaussian prior distribution, the replica method can be also used to calculate the asymptotic bit error probability using a linear minimum mean-squared error (LMMSE) detector instead of the optimum detector. However, the formulas arising from that calculation are not much faster to evaluate than to simply simulate the detector's performance. Therefore, we consider them of limited importance and omitted them here. However, they have been helpful to validate the results obtained by the replica method, since the asymptotic performance of the LMMSE can also be calculated making use of Girko's theorem [28] of large dimensional random matrices with independent non-identically distributed entries.

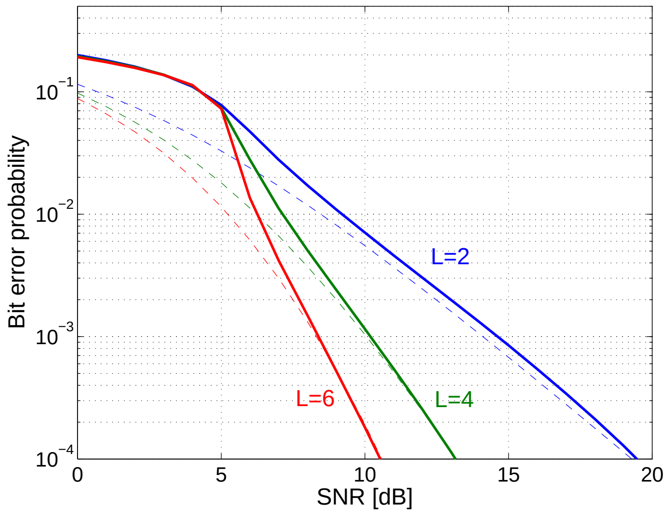

Fig. 1. Bit error probability on a Rayleigh fading channel with $L = 2, 4, 6$ paths, and load $\beta = 1.5$. The dashed-lines show the single user performance.

This latter approach was followed by Hanly and Tse [27] in the context of CDMA with macro-diversity. The results from the replica calculations match with the results obtained using Girko's theorem. This is particularly important, since the replica method makes some assumptions, such as replica symmetry and replica continuity [1], [3], [14], which are not rigorously justified in literature yet.

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