Minimum Bit Error Probability of Large Randomly Spread MC-CDMA Systems in Multipath Rayleigh Fading

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MC-CDMA in the Frequency Domain

Equivalent baseband vector channel:

$$\underset{\substack{N\times 1\\ \text{frequency}\\ \text{chips}}}{\boldsymbol{y}} = \Bigg(\underset{\substack{N\times K\\ \text{channel}\\ \text{matrix}}}{\boldsymbol{W}} \underset{\substack{\text{Hadamard}\\ \text{product}}}{\odot} \underset{\substack{N\times K\\ \text{spreading}\\ \text{matrix}}}{\boldsymbol{S}}\Bigg) \underset{\substack{K\times 1\\ \text{users'}\\ \text{data}}}{\boldsymbol{x}} + \underset{\substack{N\times 1\\ \text{noise}\\ \text{vector}}}{\boldsymbol{n}}$$

- The noise $\boldsymbol{n}$ has i.i.d. Gaussian entries with *zero-mean* and *unit variance*.
- The columns of $\boldsymbol{S}$ are the random spreading sequences of the users.
- The columns of $\boldsymbol{W}$ are the random frequency responses of the users.

Real vs. Complex Channel

In reality the channel is complex-valued.

It was shown by Tanaka ('04) that binary signaling on a real channel is equivalent in terms of large system performance to Gray-mapped QPSK on a complex channel.

Here, we use a real-valued channel with binary signaling (BPSK) to ease notation.

Prior Distribution

Let k index the user of interest.

- The data symbols of user k are equally likely $+1$ or -1.
- The data symbols of the interfering users $j \neq k$ follow the distribution

$$p_{x_j}(x_j) = \frac{1+t_j}{2}\,\delta(x_j - 1) + \frac{1-t_j}{2}\,\delta(x_j + 1), \qquad t_j \in [-1;+1].$$

Unequal priors occur when the detector is embedded into an iterative belief propagation (turbo) decoder.

Large System Limit

For i.i.d. random spreading sequences with zero mean and variance $\frac{1}{N}$, let $N, K \to \infty$, but keep the load

$$\beta \triangleq \frac{K}{N}$$

fixed.

The similarity of this limit with spin glas models in statistical physics allows for analysis by means of the replica method.

for details on applying the replica method to CDMA communication channels, see

T. Tanaka. A statistical mechanics approach to large-system analysis of CDMA multiuser detectors. *Trans. Inform. Theory*, vol. 48, no. 11, pp. 2888-2910, Nov. 2002.

Minimum Conditional Probability of Error

Maximum a-posteriori detector:

$$\hat{x}_k = \arg\max_{x_k} \Pr(x_k | \boldsymbol{y}, \boldsymbol{W})$$

In the large system limit, there is an equivalent AWGN channel with SINR $\tilde{E}_k$ such that

$$\Pr(\hat{x}_k \neq x_k | \boldsymbol{W}) = \int_{\sqrt{\tilde{E}_k}}^{\infty} Dz = \mathrm{Q}\left(\sqrt{\tilde{E}_k}\right)$$

with

$$Dz \triangleq \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

SINR of Equivalent AWGN Channel

For N, K large, solve the fixed-point system of equations

$$\tilde{E}_k = \frac{1}{N} \sum_{c=1}^{N} E_c |w_{ck}|^2$$

$$E_c = \frac{1}{\sigma_n^2 + \frac{\beta}{K} \sum_{k=1}^{K} (1 - t_k)^2 |w_{ck}|^2 \int \frac{1 - \tanh\left(z\sqrt{\tilde{E}_k} + \tilde{E}_k\right)}{1 - t_k^2 \tanh^2\left(z\sqrt{\tilde{E}_k} + \tilde{E}_k\right)} Dz}$$

In case of multiple solutions, see paper.

Unconditional Probability of Error

$$\Pr(\hat{x}_k \neq x_k) = \underset{\boldsymbol{W}}{\mathrm{E}} \Pr(\hat{x}_k \neq x_k | \boldsymbol{W}) = \underset{\boldsymbol{W}}{\mathrm{E}} \, \mathrm{Q}\left(\sqrt{\tilde{E}_k}\right)$$

This expectation can be found by simulations for arbitrary joint statistics of $\boldsymbol{W}$.

In practice, the fading statistics obey some structure:

- Asymptotic frequency-invariance on the uplink (reverse link)
- Rank-1 statistics on the downlink (forward link)

Asymptotic Frequency Invariance

$$E_c = E \qquad \forall c$$

The fading is ergodic across the user population for each frequency c.

$$\tilde{E}_k = \frac{P_k}{\sigma_n^2 + \frac{\beta}{K}\sum_{n=1}^{K}(1-t_n)^2 P_n \int \frac{1-\tanh\left(z\sqrt{\tilde{E}_n}+\tilde{E}_n\right)}{1-t_n^2\tanh^2\left(z\sqrt{\tilde{E}_n}+\tilde{E}_n\right)} Dz}$$

with

$$P_k = \frac{1}{N}\sum_{c=1}^{N}|w_{ck}|^2$$

The spectrum of the received signal is white (frequency-invariant).
Full diversity is achieved.

Rank 1 Statistics

$$\boldsymbol{W} = \boldsymbol{f}\boldsymbol{u}^{\mathrm{T}} \qquad \Longleftrightarrow \qquad w_{ck} = f_c u_k$$

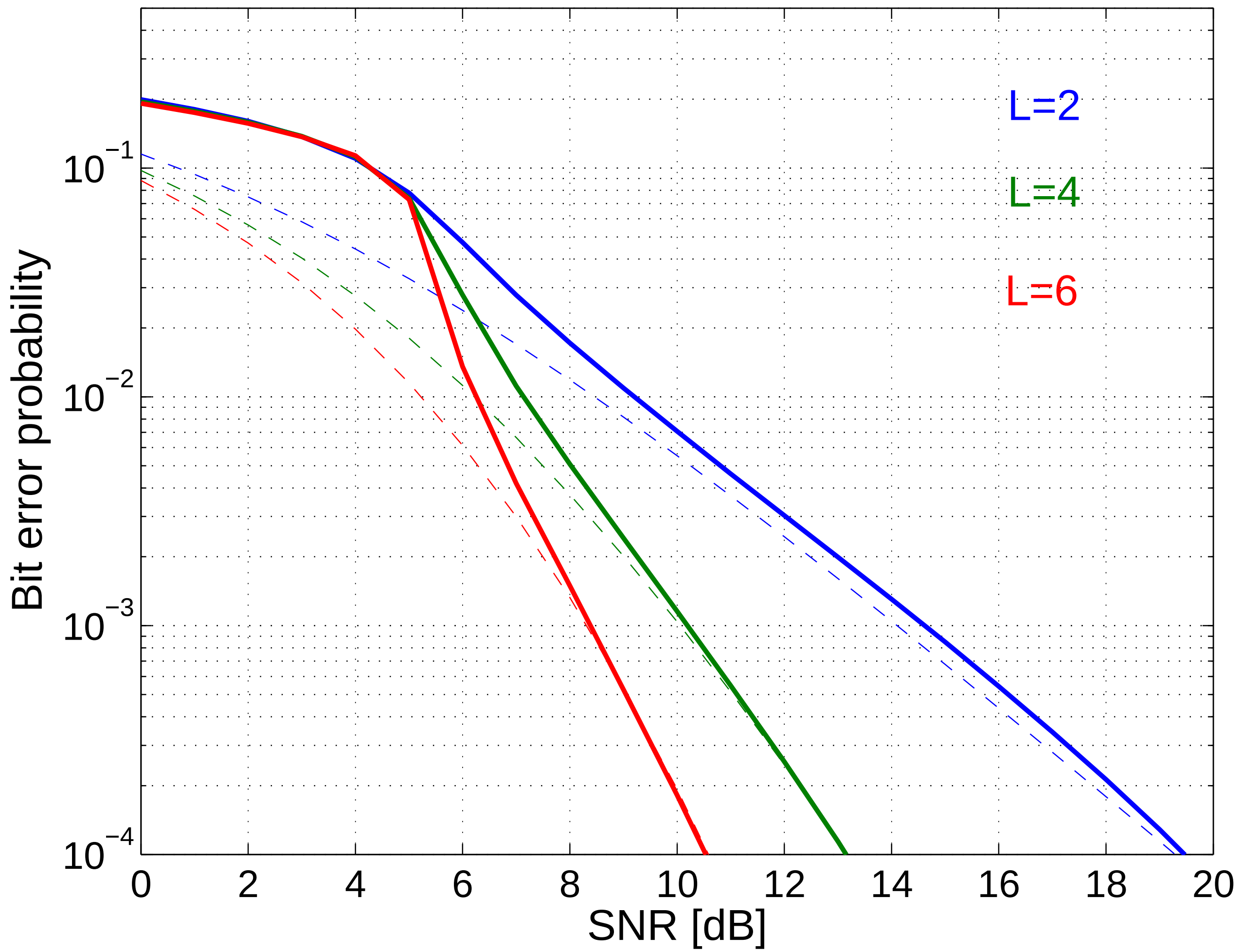
All users experience the same fading channel except for a scalar factor u_k.

$$\tilde{E}_k = \frac{|u_k|^2}{N} \sum_{c=1}^{N} \frac{1}{\frac{\sigma_n^2}{|f_c|^2} + \frac{\beta}{K} \sum_{n=1}^{K} (1-t_n)^2 |u_n|^2 \int \frac{1-\tanh\left(z\sqrt{\tilde{E}_n} + \tilde{E}_n\right)}{1 - t_n^2 \tanh^2\left(z\sqrt{\tilde{E}_n} + \tilde{E}_n\right)} Dz}$$

Full diversity is achieved.

The spectrum of the received signal is colored $\Longrightarrow$ degradation.

Uplink

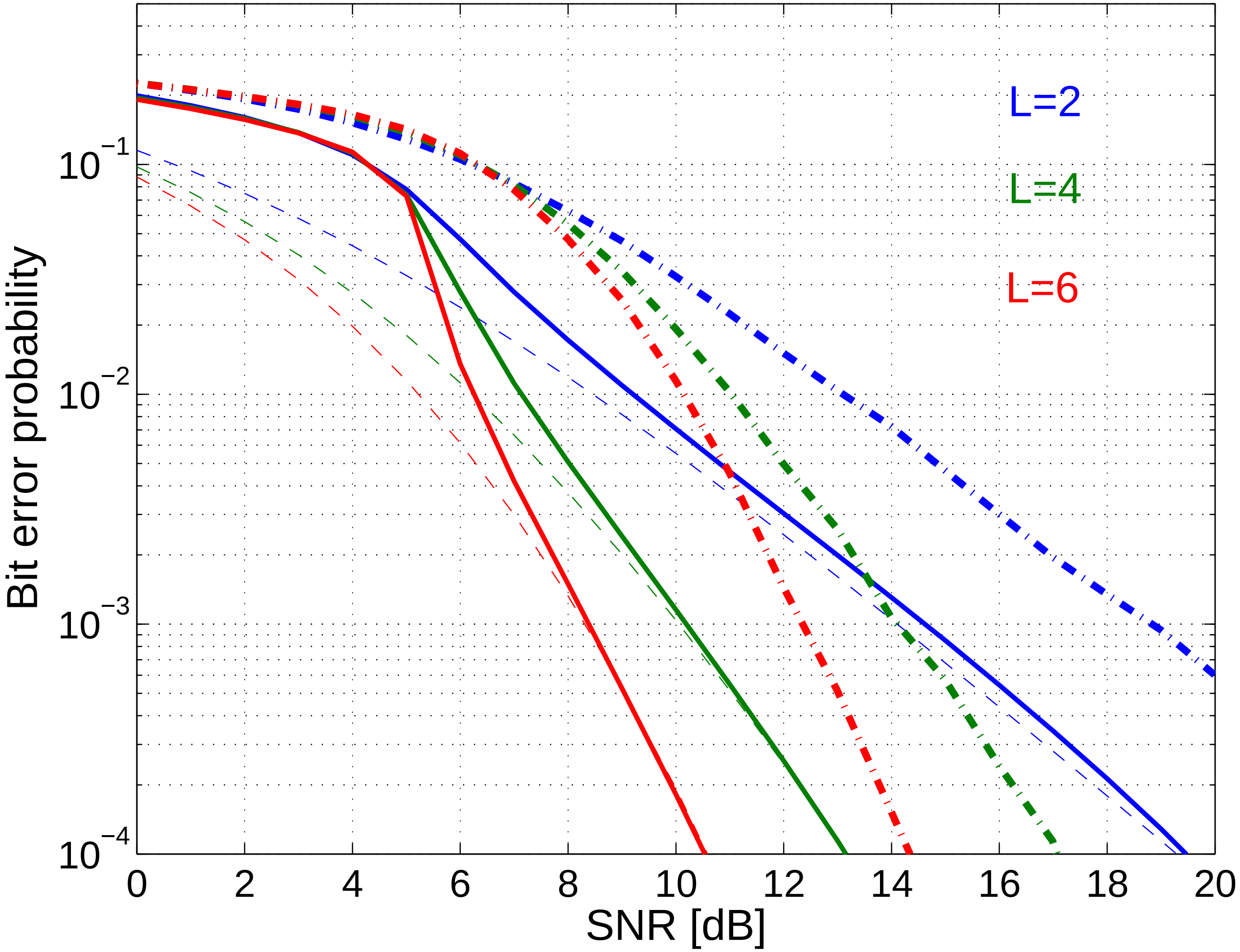


Uniform priors

L equal power paths

$$\frac{K}{N} = 1.5$$

Uplink vs. Downlink



Uniform priors

L equal power paths

$$\frac{K}{N} = 1.5$$

4 dB difference!

Results

- Due to asymptotic frequency invariance, the single user bound (SUB) is achieved in the uplink.

- Due to rank 1 statistics, the SUB is *NOT* achieved in the downlink.

In practice, the downlink employs orthogonal or WBE sequences. Nevertheless, the SUB cannot be achieved.