On Spectral Efficiency of Vector Precoding for Gaussian MIMO Broadcast Channels

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***Abstract*—The spectral efficiency of practically oriented vector precoding schemes for the Gaussian multiple-input multiple-output (MIMO) broadcast channel is analyzed in the large system limit. Considering discrete complex input alphabets, the transmitter is assumed to comprise a linear front-end combined with nonlinear precoding, that minimizes the transmit energy penalty imposed by the linear front-end. The energy penalty is minimized by relaxing the input alphabet to a larger alphabet set prior to precoding. The so-called "replica method" of statistical physics is employed to derive the limiting empirical distribution of the precoder's output, as well as the limiting energy penalty. Particularizing to a "zero-forcing" (ZF) linear front-end, and non-cooperative users, a decoupling result is derived according to which the channel observed by each of the individual receivers can be characterized by the Markov chain u–x–y, where u is the channel input, x is the equivalent precoder output, and y is the channel output. A comparative spectral efficiency analysis of two illustrative examples reveals *significant performance gains* compared to linear ZF precoding in the medium to high E_b/N_0 region. In particular, we demonstrate that *convex* extended alphabets, amenable to efficient energy minimization algorithms, provide an attractive alternative to alphabets based on the discrete Gaussian integer lattice, for which the energy minimization problem is NP-hard.**

I. Introduction

Consider a wireless MIMO broadcast channel (BC) setting, where the transmitter has N transmit antennas and the K users have single receive antennas. No user-cooperation of any kind is assumed. The received signals are embedded in additive white Gaussian noise (AWGN). The setting falls within the framework of the Gaussian MIMO broadcast channel, whose capacity region is now well known to be the *dirty paper coding* (DPC) [1] capacity region [2]. See [3] for a recent survey on information theoretic aspects of MIMO broadcast channels.

Due to the complexity of DPC, simpler practically oriented suboptimum transmitter preprocessing schemes are appealing. One such scheme is channel inversion, or *"zero-forcing"* (ZF) at the transmitter (e.g., [4]), however it performs poorly when the channel transfer matrix has small singular values. To overcome this deficiency, several lattice-based precoding schemes have been proposed, e.g., [5]–[7]. Particularly, a *vector perturbation* technique was introduced in [6], based on the idea of Tomlinson-Harashima precoding (see therein). Accordingly, a scaled complex integer vector is added to each data vector, chosen to minimize the energy penalty of the ZF front-end. A modulo function is employed at the receivers, uniquely determining the transmitted symbols in the absence of noise. A related approach can also be found, e.g., in [7] (see references therein for additional literature in this framework), where lattice-basis reduction techniques are employed to avoid the complexity of solving the K-dimensional integer-lattice least squares problem required by the perturbation method. So far these schemes have been analyzed mainly in terms of uncoded symbol error probability (via simulations), asymptotic capacity scaling laws and diversity order (the asymptotic slope of the error probability in the high signal-to-noise ratio (SNR) regime), and Monte-Carlo simulations to obtain information theoretic achievable rates (see [8] for a semi-tutorial review in this respect). The full effect of such schemes on *coded* communication in all SNRs is yet to be explored.

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In this paper we consider a novel nonlinear precoding approach recently proposed by Müller *et al.* [9]. Full channel state information (CSI) is assumed available at the transmitter, while the receivers are cognizant of their own channels only (more on this later). The transmitter comprises a linear front-end combined with nonlinear precoding. The nonlinear part relies on relaxation of the transmitted symbols' alphabets to larger alphabet sets. The idea is to optimize the vector of transmitted symbols over the extended alphabet set, so as to minimize the energy penalty imposed by the linear front-end. A notable feature of this precoding scheme is that it can be combined with *convex* extended alphabet sets, lending themselves to *efficient* practical energy minimization algorithms. Müller *et al.* [9] mainly focus on presenting the method, and on the derivation of the energy penalty in the asymptotic regime, in which both the number of transmit antennas N and the number of users K go to infinity, while $K/N \to \alpha < \infty$ (α is commonly referred to as the *system load*). Here we take one step further, and provide an information theoretic perspective of this precoding approach by considering coded transmissions and achievable throughputs. The employed performance measure is the normalized spectral efficiency, defined as the total number of bits/sec/Hz *per transmit antenna* that can be transmitted arbitrarily reliably through the broadcast channel.

The key tool for the analysis, as in [9], is the so-called replica method. The replica method was originally invented for the analysis of spin glasses in statistical physics, and has been recently successfully applied to problems in wireless communications and coding theory. It has also been recognized by now as an important tool for information theoretic analyses in cases where "conventional" random matrix theory does not apply. Although the replica method is heuristic in nature, extensive simulations and exact analytical results in the literature suggest that the replica analysis generally yields excellent approximations in many cases of interest (see references in [9] and also [10] for a recent tutorial manuscript).

Applying the replica method, the limiting marginal conditional distribution of the nonlinear precoder's output, as well as the limiting energy penalty, are *analytically formulated*. It is noted, however, that the validity of these asymptotic results as approximations for practical systems of finite size is very much dependent on the underlying assumptions imposed during the replica analysis. In particular, the results in [9] are based on what is known in the statistical physics literature as the *replica symmetry* (RS) assumption (see therein). While this assumption is known to be generally adequate for analyzing *convex* optimization problems (see, e.g., supportive simulation results in [11]), it turns out that, at least in the current setting, the RS assumption provides a rather loose approximation for non-convex extended alphabets, especially as the system load gets close to unity. This is observed by comparing the energy penalty to an asymptotic lower bound recently obtained using lattice-theoretic arguments in [12]. Therefore, we provide in this paper novel results based on what is referred to as the *first order replica symmetry breaking* (1RSB) assumption [13], to more accurately describe nonconvex extended alphabet settings. Using either type of results, while focusing on a ZF front-end with load $\alpha \leq 1$, the spectral efficiency can be analytically expressed via the input-output mutual information of the single-user channel observed by each of the receivers (assumed aware of the channel's statistics by our main results). Two illustrative particular cases are discussed, focusing on a quadrature phase shift keying (QPSK) input. The results indicate *significant* performance enhancement over linear ZF preprocessing in the medium to high E_b/N_0 region.

The structure of the rest of this paper is as follows. Section II describes the system model. Section III states the main analytical results under the RS and 1RSB assumptions. Section IV discusses particular examples of extended alphabet sets. Section V presents some numerical results. Finally, Section VI ends this paper with some concluding remarks.

II. System Model

Consider the following Gaussian MIMO broadcast channel model:

$$\boldsymbol{r} = \boldsymbol{H}\boldsymbol{t} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{r}_{[K\times 1]}$ is the vector of received signals, $\boldsymbol{H}_{[K\times N]}$ is the channel transfer matrix, assumed to be of unit row norm, $\boldsymbol{t}_{[N\times 1]}$ is the vector of transmitted signals, and $\boldsymbol{n}_{[K\times 1]}$ is the vector of i.i.d. zero mean proper complex AWGNs at the users' receivers. We denote the noises' spectral level by σ^2 so that $\boldsymbol{n} \sim \mathcal{N}_c(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$. Let $\boldsymbol{u}_{[K\times 1]}$ denote the vector of encoders' outputs destined to each user. The actually transmitted vector $\boldsymbol{t}$ is related to $\boldsymbol{u}$ through the following nonlinear transformation

$$\boldsymbol{t} = \frac{\boldsymbol{T}\boldsymbol{x}(\boldsymbol{T},\boldsymbol{u})}{\|\boldsymbol{T}\boldsymbol{x}(\boldsymbol{T},\boldsymbol{u})\|} \triangleq \frac{\boldsymbol{T}\boldsymbol{x}(\boldsymbol{T},\boldsymbol{u})}{\sqrt{\mathscr{E}^{\mathsf{tot}}(\boldsymbol{T},\boldsymbol{x})}} \quad , \tag{2}$$

where $\boldsymbol{T}_{[N\times K]}$ is the front-end linear precoding matrix, $\boldsymbol{x}(\boldsymbol{T},\boldsymbol{u})$ is a vector chosen from an extended alphabet to minimize the effective instantaneous transmit energy penalty, and $\mathscr{E}^{\mathsf{tot}}(\boldsymbol{T},\boldsymbol{x})$ denotes the energy penalty induced by the precoding matrix $\boldsymbol{T}$ and the particular choice of $\boldsymbol{x}$ (the explicit dependence on the arguments is omitted henceforth for simplicity). The transmitted vector $\boldsymbol{t}$ is thus normalized to have a unit instantaneous energy, and we define the transmit SNR as $\mathsf{snr} \triangleq 1/(K\sigma^2)$. To be more specific, let $\mathscr{U}$ denote the *discrete* alphabet of the coded symbols, i.e., $\boldsymbol{u} = [u_1, \ldots, u_K]^T \in \mathscr{U}^K$. These symbols are assumed to be random variables, independent across users, and subject to an identical underlying distribution $P_u(u)$ (we use henceforth the notation $dP_u(u) \triangleq f_u(u)\,du \triangleq \sum_{u_i\in\mathscr{U}} P_u(u_i)\delta(u-u_i)\,du$). The relaxed (extended) alphabet $\mathscr{B}$ is a union of disjoint sets $\mathscr{B}_u$, $u \in \mathscr{U}$, so that every coded symbol $u \in \mathscr{U}$ can be represented using any element of $\mathscr{B}_u$ *without ambiguity* [9]. The vector $\boldsymbol{x} = [x_1, \ldots, x_K]^T$ thus satisfies

$$\boldsymbol{x} = \underset{\boldsymbol{x}\in\mathscr{B}_{u_1}\times\cdots\times\mathscr{B}_{u_K}}{\operatorname{argmin}} \|\boldsymbol{T}\boldsymbol{x}\|^2 \quad . \tag{3}$$

III. Summary of the Main Results

A. General Results: RS Assumption

We first review the key result of [9], which contains some of the central ingredients of the main results of this paper.

Proposition 3.1: Suppose the random matrix $\boldsymbol{J} = \boldsymbol{T}^\dagger\boldsymbol{T}$ can be decomposed as

$$\boldsymbol{J} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger \tag{4}$$

where $\boldsymbol{D}$ is diagonal and $\boldsymbol{U}$ is Haar distributed [14]. Moreover, as $K \to \infty$, the empirical distribution of the diagonal elements of $\boldsymbol{D}$ (i.e., the eigenvalues of $\boldsymbol{J}$) converges to a nonrandom distribution uniquely characterized by its R-transform[1] $R(\cdot)$. Then under some technical assumptions, including in particular *replica symmetry* and a self averaging property (see [9]), the effective energy penalty per symbol

$$\begin{aligned} \frac{1}{K}\mathscr{E}^{\mathsf{tot}} \;&=\; \frac{1}{K}\min_{\boldsymbol{x}\in\mathscr{B}_{u_1}\times\cdots\times\mathscr{B}_{u_K}} \boldsymbol{x}^\dagger\boldsymbol{J}\boldsymbol{x} \\ &\longrightarrow q_0[R(-\chi_0) - \chi_0 R'(-\chi_0)] \triangleq \bar{\mathscr{E}}_{\mathsf{rs}} \end{aligned} \tag{5}$$

in probability, as $K, N \to \infty$, $K/N \to \alpha < \infty$. Here $R'(\cdot)$ denotes the derivative of the function $R(\cdot)$, $\chi_0 \in (0,\infty)$, and the parameters q_0 and χ_0 satisfy the following pair of coupled

[1] For an arbitrary distribution $P(x)$, let $m(s) = \int \frac{dP(x)}{x-s}$ denote its Stieltjes transform. Then, the R-transform of $P(x)$ is $R(w) = m^{-1}(-w) - \frac{1}{w}$, where $m^{-1}(\cdot)$ denotes the inverse function of $m(\cdot)$.

fixed-point equations:

$$q_0 = \int\int_{\mathbb{C}} \left| \underset{x\in\mathscr{B}_u}{\operatorname{argmin}} \left| z - \frac{R(-\chi_0)\, x}{\sqrt{q_0 R'(-\chi_0)}} \right| \right|^2 Dz\, dP_u(u) \ , \tag{6}$$

$$\chi_0 = \frac{\int \Re\{\int_{\mathbb{C}} \underset{x\in\mathscr{B}_u}{\operatorname{argmin}} \left| z - \frac{R(-\chi_0)\, x}{\sqrt{q_0 R'(-\chi_0)}} \right| z^*\, Dz\}\, dP_u(u)}{\sqrt{q_0 R'(-\chi_0)}} \ , \tag{7}$$

where $\int_{\mathbb{C}}(\cdot)Dz \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\cdot)\frac{\mathrm{e}^{-|z|^2}}{\pi}dz_r dz_i$, for $z \triangleq z_r + jz_i$, and $\Re\{\cdot\}$ in (7) takes the real part of the inner integral's result.

The main result of this section is given next.

Proposition 3.2: With the same underlying assumptions and notation as in Proposition 3.1, the limiting empirical distribution of the nonlinear precoder's outputs given an input symbol u satisfies

$$P_{x|u}(\tilde{x}|u) = \Pr\{x = \tilde{x} \in \mathscr{B}_u | u\} = \int_{\mathbb{C}} 1\left\{ \tilde{x} = \underset{x\in\mathscr{B}_u}{\operatorname{argmin}} \left| z - \frac{R(-\chi_0)x}{\sqrt{q_0 R'(-\chi_0)}} \right| \right\} Dz \ , \tag{8}$$

where $1\{\cdot\}$ is the indicator function. This is the measure of the Voronoi region in the scaled *conditional* signal constellation $\mathscr{B}_u$, with respect to the (complex) Gaussian measure.

Proof: The proof follows similar steps to those in [9, Appendix A], while considering the empirical joint moments of x and u (see [15]) and the corresponding characteristic function. The reader is referred to [16] for the full details. ■

For the particular case of $\boldsymbol{H}$ having i.i.d. zero-mean circularly symmetric complex Gaussian entries with variance $\frac{1}{N}$ ("a Gaussian $\boldsymbol{H}$"), the above results can be considerably simplified by observing that $\alpha\bar{\mathscr{E}}_{\mathsf{rs}} = q_0 R'(-\chi_0)/R^2(-\chi_0)$ [9]. The energy penalty and the conditional distribution are thus obtained while solving only a *single* fixed-point equation.

B. General Results: 1RSB Assumption

This section presents analogous results to Section III-A, while assuming 1RSB. The proofs are given in [16]. The 1RSB results are specified in terms of four macroscopic parameters $q_1, p_1, \chi_1, \mu_1 \in (0, \infty)$. For $y, z \in \mathbb{C}$, let

$$\mathscr{F}_u(y,z) \triangleq \mathrm{e}^{-\mu_1\left(\min\limits_{x\in\mathscr{B}_u} \varepsilon_1|x|^2 - 2\Re\{x(f_1 z^* + g_1 y^*)\}\right)} \ , \tag{9}$$

where the parameters ε_1, g_1 and f_1 are defined as

$$\varepsilon_1 = R(-\chi_1) \ , \tag{10}$$

$$g_1 = \sqrt{(R(-\chi_1) - R(-\chi_1 - \mu_1 p_1))/\mu_1} \ , \tag{11}$$

$$f_1 = \sqrt{q_1 R'(-\chi_1 - \mu_1 p_1)} \ . \tag{12}$$

Then $\{q_1, p_1, \chi_1, \mu_1\}$ are the solutions to the four equations

$$\chi_1 + p_1\mu_1 = \frac{1}{f_1}\int\int_{\mathbb{C}^2} \Re\left\{ z^* \underset{x\in\mathscr{B}_u}{\operatorname{argmin}} |f_1 z + g_1 y - \varepsilon_1 x| \right\} \cdot \frac{\mathscr{F}_u(y,z)}{\int_{\mathbb{C}} \mathscr{F}_u(y,z)\, Dy} Dy\, Dz\, dP_u(u) \ , \tag{13}$$

$$\chi_1 + (q_1 + p_1)\mu_1 = \int\int_{\mathbb{C}^2} \Re\left\{ y^* \underset{x\in\mathscr{B}_u}{\operatorname{argmin}} |f_1 z + g_1 y - \varepsilon_1 x| \right\} \cdot \frac{\mathscr{F}_u(y,z)}{g_1 \int_{\mathbb{C}} \mathscr{F}_u(y,z)\, Dy} Dy\, Dz\, dP_u(u) \ , \tag{14}$$

$$q_1 + p_1 = \int\int_{\mathbb{C}^2} \left| \underset{x\in\mathscr{B}_u}{\operatorname{argmin}} |f_1 z + g_1 y - \varepsilon_1 x| \right|^2 \cdot \frac{\mathscr{F}_u(y,z)}{\int_{\mathbb{C}} \mathscr{F}_u(y,z)\, Dy} Dy\, Dz\, dP_u(u) \ , \tag{15}$$

and

$$\begin{aligned}\int_{\chi_1}^{\chi_1+\mu_1 p_1} R(-w)\, dw = \\ \int\int_{\mathbb{C}} \log\left(\int_{\mathbb{C}} \mathscr{F}_u(y,z)\, Dy \right) Dz\, dP_u(u) - 2\chi_1 R(-\chi_1) \\ +(\mu_1 q_1 + 2\chi_1 + 2\mu_1 p_1) R(-\chi_1 - \mu_1 p_1) \\ -2\mu_1 q_1 (\chi_1 + \mu_1 p_1) R'(-\chi_1 - \mu_1 p_1) \ .\end{aligned} \tag{16}$$

Proposition 3.3: Suppose the same conditions and assumptions as in Proposition 3.1 hold, except that 1RSB is assumed instead of RS. Then, the effective energy penalty per symbol

$$\begin{aligned}\frac{1}{K}\mathscr{E}^{\mathsf{tot}} \longrightarrow \left(q_1 + p_1 + \frac{\chi_1}{\mu_1} \right) R(-\chi_1 - \mu_1 p_1) - \frac{\chi_1}{\mu_1} R(-\chi_1) \\ -q_1(\chi_1 + \mu_1 p_1) R'(-\chi_1 - \mu_1 p_1) \triangleq \bar{\mathscr{E}}_{\mathsf{rsb1}}\end{aligned} \tag{17}$$

in probability, as $K, N \to \infty$, $K/N \to \alpha < \infty$.

Proposition 3.4: With the same underlying assumptions and notation as in Proposition 3.3, the limiting empirical distribution of the nonlinear precoder's outputs given an input symbol u satisfies

$$P_{x|u}(\tilde{x}|u) = \int_{\mathbb{C}}\int_{\mathbb{C}} 1\left\{ \tilde{x} = \underset{x\in\mathscr{B}_u}{\operatorname{argmin}} |f_1 z + g_1 y - \varepsilon_1 x| \right\} \cdot \frac{\mathscr{F}_u(y,z)}{\int_{\mathbb{C}} \mathscr{F}_u(y,z)\, Dy} Dy\, Dz \ . \tag{18}$$

C. Zero-Forcing Front-End

With a ZF front-end, the precoding matrix $\boldsymbol{T}$ is given by the pseudo-inverse of the channel transfer matrix, which we write here as

$$\boldsymbol{T} = \boldsymbol{H}^+ = \boldsymbol{H}^\dagger \left(\boldsymbol{H}\boldsymbol{H}^\dagger \right)^{-1} \ . \tag{19}$$

The underlying assumptions are that $N \geq K$ and that the matrix $\boldsymbol{H}\boldsymbol{H}^\dagger$ is a.s. positive definite (which applies for the case of a Gaussian $\boldsymbol{H}$). Focusing on the asymptotic regime for $K/N \to \alpha \leq 1$, then using (1), (2), and either Propositions 3.1 or 3.3, the equivalent single-user channel observed by user i can be represented as

$$\check{r}_i \approx x_i + \check{n}_i, \quad K \gg 1 \ , \tag{20}$$

where $\check{n}_i$ is a zero mean circularly symmetric complex Gaussian noise with variance $1/\rho$, $\rho \triangleq \mathsf{snr}/\bar{\mathscr{E}}$ denotes the effective received SNR, and $\bar{\mathscr{E}}$ is given by either (5) or (17).

Proposition 3.5: Employing the same underlying assumptions as in Propositions 3.1 or 3.3, then with a ZF front-end the channel observed by a randomly chosen user is equivalent to a concatenated single-user channel, with input $u \in \mathscr{U}$, intermediate output $x \in \mathscr{B}_u$ and final output $y \in \mathbb{C}$, specified by the Markov chain u–x–y. This Markov chain is defined by the following joint probability density function

$$f_{uxy}(u,x,y) = f_u(u) f_{x|u}(x|u) f_{y|x}(y|x) \quad , \tag{21}$$

where $f_{x|u}(x|u) = \sum_{x_i \in \mathscr{B}_u} P_{x|u}(x_i|u)\delta(x - x_i)$, with $P_{x|u}(x_i|u)$ given by (8) or (18), and $f_{y|x}(y|x) = \frac{\rho}{\pi}\mathrm{e}^{-|y-x|^2\rho}$ is the Gaussian density with mean x and variance $1/\rho$.

Proof: The Proposition follows straightforwardly from Propositions 3.2 or 3.4, and (20). ■

The achievable throughput of the proposed scheme can be derived in terms of the equivalent single user channel using Proposition 3.5. This throughput is given by the mutual information[2] between the input u and received signal y, i.e.,

$$R = I(u;y) = H(y) - H(y|u) \quad , \tag{22}$$

where $H(\cdot)$ and $H(\cdot|\cdot)$ denote entropy and conditional entropy, respectively (which can be readily calculated using Proposition 3.5). The spectral efficiency is then given by

$$C \approx \frac{K}{N} R \xrightarrow[K \to \infty]{} \alpha R \quad , \tag{23}$$

and it is functionally dependent on the system average $\frac{E_b}{N_0}$ through the relation $\mathsf{snr} = \frac{1}{\alpha} C \frac{E_b}{N_0}$ [17].

To get a better insight into the impact of the nonlinear precoding scheme, it is useful to compare the results to the spectral efficiency of DPC with Gaussian input (specifying the ultimate performance), as well as to the spectral efficiency of *linear* ZF (for both Gaussian and discrete alphabet input). The DPC spectral efficiency coincides for a Gaussian $\boldsymbol{H}$ with the spectral efficiency of the optimum receiver in the *dual uplink* channel with uniform power distribution [17] (see [3] and justification in [18]). For linear ZF precoding, the induced energy penalty at the large system limit is $1/(1-\alpha)$, and again for Gaussian input the spectral efficiency coincides with the corresponding result in [17] (see also [4]). The corresponding spectral efficiency with discrete input alphabets can be straightforwardly derived in an analogous manner.

IV. Particular Examples

Adhering to [9], we briefly describe in the following two particular examples of extended alphabet sets for QPSK signaling (translating to $\mathscr{U} = \{1+j, -1+j, -1-j, 1-j\}$). Quadrature symmetric transmissions are assumed.

The first example corresponds to the case in which the relaxed alphabets can be represented as points from the extended lattice $\mathscr{B}_u = \frac{u}{1+j}((4\mathbb{Z}+1)\times(4\mathbb{Z}+1))$, $\forall u \in \mathscr{U}$. More specifically, $\mathscr{B}_{\pm 1 \pm j} = \pm\{c_1, c_2, \ldots, c_L\} \pm j\{c_1, c_2, \ldots, c_L\}$, where $-\infty < c_1 < \cdots < c_L < \infty$. The parameter L thus denotes the number of lattice points used in the extended alphabet in each dimension, and we particularize here to the set $\{+1, -3, +5, -7, +9, \ldots\}$. The tools used to analyze the performance of this alphabet relaxation scheme are Propositions 3.3 – 3.5. The validity of the 1RSB assumption in this setting is supported by numerical simulations [16]. The RS assumption is found to be too optimistic, and it predicts energy penalties up to 2.8dB lower close to unit load. The energy penalty given by Proposition 3.3 exceeds the lower bound of [12] for $\alpha \in (0,1)$, and coincides with the bound as $\alpha \to 1$.

[2]The underlying assumption here is that the receiver is fully aware of the channel statistics in view of the limiting results of Propositions 3.1 – 3.4.

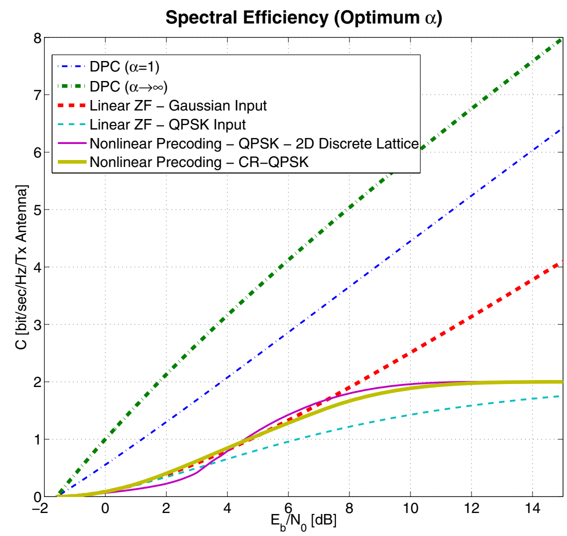

Fig. 1. Spectral efficiency results optimized with respect to the load α.

Applying a continuity argument, we consider next the following relaxation for the QPSK constellation: $\mathscr{B}_{1+j} = \{z \in \mathbb{C} : \Re\{z\} \geq 1, \Im\{z\} \geq 1\}$, and $\mathscr{B}_u = \frac{u}{1+j}\mathscr{B}_{1+j}$ for $u \in \{-1+j, -1-j, 1-j\}$. These four sets are independent (assuming independent input symbols), and *convex*, allowing for an efficient solution to the corresponding quadratic programming problem of minimizing the energy penalty. Furthermore, the optimization problem is also replica symmetric [11] and can be analyzed using Propositions 3.1, 3.2 and 3.5. This transmission scheme is henceforth referred to as *convex relaxation QPSK (CR-QPSK)*.

V. Numerical Results

Comparative spectral efficiency results are plotted in Fig. 1. The figure shows the spectral efficiency with nonlinear precoding for QPSK while taking $L = 2$ for the discrete extended alphabet ($L > 2$ provides negligible improvement for $\alpha \leq 1$ [16]), as well as the spectral efficiency of linear ZF precoding for Gaussian and QPSK input. The spectral efficiencies were evaluated for the *optimum* choice of the system load $\alpha \in (0,1]$, which is in general a function of E_b/N_0. The DPC spectral efficiency is also provided for comparison, evaluated both for $\alpha = 1$, and at the limit as $\alpha \to \infty$ (specifying the ultimate performance). The optimization with respect to α emphasizes its role as a crucial system design parameter, facilitating the proper working point for each transmission scheme, per each E_b/N_0. Note also that this optimization has a natural practical scheduling interpretation, by specifying the desired number of simultaneously active scheduled users per transmit antenna (see, e.g., [18]). A Gaussian $\boldsymbol{H}$ is assumed in all cases.

The results indicate that nonlinear precoding can provide significant performance enhancement for medium to high E_b/N_0 values. The two-dimensional discrete relaxation scheme for QPSK is shown to outperform linear ZF with QPSK input for $E_b/N_0 > 3.34$dB. The performance enhancement becomes more pronounced the more the spectral efficiency approaches the upper limit of 2 bits/sec/Hz per transmit antenna. For example, a spectral efficiency of 1.75 bits/sec/Hz can be obtained with the nonlinear scheme already at $E_b/N_0 \approx 7.66$dB, whereas linear ZF requires additional 7.26dB for the same spectral efficiency. In fact, the QPSK-based nonlinear precoding scheme is shown to marginally outperform linear ZF with Gaussian input for $4.17\text{dB} < E_b/N_0 < 7.26\text{dB}$. The gap from the DPC upper bound is however still essentially retained (4.49dB at 1.75 bits/sec/Hz, considering DPC with $\alpha = 1$, to make a more fair comparison).

Turning to the CR-QPSK scheme, Fig. 1 shows that it also provides a considerable performance enhancement over linear ZF with QPSK input. It is outperformed by the discrete relaxation scheme for $E_b/N_0 > 4.37$dB (somewhat surprisingly it performs better at lower values of E_b/N_0, and in fact it even negligibly outperforms linear ZF with *Gaussian* input in the low E_b/N_0 region), however the performance enhancement of the discrete scheme is relatively small (only 0.94dB at 1.75 bits/sec/Hz). Moreover, unlike the discrete scheme, CR-QPSK outperforms linear precoding (with QPSK input) for *all* E_b/N_0 values. These results are of particular interest as the CR-QPSK scheme lends itself to efficient implementation, whereas the discrete relaxation scheme involves the solution of an NP-hard optimization problem. The reader is referred to [16] for a more detailed numerical analysis.

VI. Concluding Remarks

Nonlinear vector precoding for the MIMO broadcast channel, based on relaxation of the input alphabet, was investigated in this paper in the large system limit using the replica analysis of statistical physics. The scheme was shown to significantly enhance performance, as compared to linear precoding. The results provide a better understanding of the impact of nonlinear precoding on *coded* communication by precisely characterizing the statistical behavior of the precoder's output, given the input, as observed by a randomly chosen user receiver. Focusing on a linear ZF front-end, the equivalent *nonlinear* channel observed by each user can be described by means of a Markov chain. This characterization captures the tradeoff between the beneficial effect of nonlinear precoding, i.e., reducing the energy penalty of the linear processing front-end, and the negative impact of the extra randomness induced at each user's receiver (recall the data processing inequality). Examples of extended alphabet sets were shortly presented, demonstrating the potential of the proposed scheme. Convex alphabet sets, as for example CR-QPSK, are practically appealing in this respect, as they are shown to exhibit an excellent performance vs. complexity tradeoff. Additional examples for extended alphabets are currently investigated, noting however that the problem of finding the optimum precoding scheme that maximizes the spectral efficiency is not at all trivial, as the corresponding equivalent channel statistics depend in this setting on the choice of input distribution and extended alphabet sets. Extension of the results to the case in which $\alpha > 1$ is also currently addressed, as well as generalizations of the linear front-end matrix.

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