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Gaussian Multiple–Access Channels with Weighted Energy Constraint

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Abstract — **The Gaussian multiple–access channel with weighted energy constraint $\tilde{E}_{\rm sum} = \sum_k E_k/|d_k|^2$ models the situation in cellular multiuser communication more accurately than the standard approach with $d_k = 1, \forall k$. The minimum weighted sum energy $\tilde{E}_{\rm sum}$ is shown to be only achievable at a unique energy tupel $(E_1, \dots, E_K)$, if all weights differ in amplitude. Moreover, the theory is generalized to a multiple receiver system describing cellular communication systems with optimum multiple cell–site processing. While for single-receiver systems the optimum energy tupel is a vertex of the allowed energy region, i.e. successive cancellation is possible, for a multi-receiver system this does not hold in general.**

I. Introduction

Recently, many publications deal with the Shannon capacity of cellular mobile radio systems, e.g. [1, 2, 3] and references therein. These models are more or less accurate, as they have to involve a trade–off between analytical tractability and a precise description of the real physical channel.

Recent literature about multiple–access communication usually aims to maximize the achievable rate for a given signal–to–noise ratio at receiver site whereas system designers intend to minimize the transmitted signal power required to maintain error free transmission at a given rate. Although for single user communication both goals are obviously equivalent, considering multiuser communication they are different aims in general. For the special case of the Gaussian multiple–access channel we show in Section II that equivalence holds, if and only if signal attenuations of all users are equal in amplitude. In Section III the channel model introduced by Wyner [2] providing multiple cell–site processing is analyzed with respect to weighted energy constraints. Section IV points out some practical implications.

II. Gaussian Multiple–Access Channel

In this section we consider a single cell system with one receiver having to support the reliable transmission of K users' data symbols at given rates R_k, $1 \le k \le K$, in bits per symbol. The transmission of the $k^{\rm th}$ user's complex Gaussian distributed code symbol sequence $\tilde{X}_k$ is interferred by complex additive white Gaussian noise with power spectral density N_0.

$$Y = \sum_{k=1}^{K} d_k X_k + N$$

Further, we assume that each user's transmit signal is attenuated by $d_k \in \mathbb{C}$, $1 \le k \le K$, which yields $E_k = \tilde{E}_k |d_k|^2$, where E_k and $\tilde{E}_k$ denote the $k^{\rm th}$ user's received and transmitted energy per symbol, respectively. The cost function having to be minimized is the total transmitted energy per symbol

$$\tilde{E}_{\rm sum} \triangleq \sum_{k=1}^{K} \frac{E_k(R_1, \dots, R_K)}{|d_k|^2}.$$

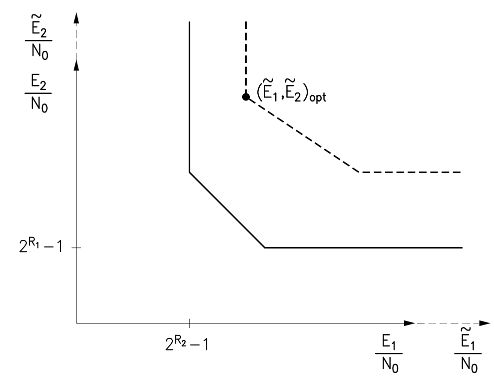

Fig. 1: Transmitted and received energy region for given rate tupel (R_1, R_2), respectively.

Theorem 1 *The minimum total transmitted energy for a given rate tupel $(R_1, \dots, R_K), R_k > 0, \forall k$, and given tupel of attenuations $(d_1, \dots, d_K)$ with $|d_i| \le |d_k|, \forall i < k$, is achieved by a the transmitted energy tupel $(\tilde{E}_1, \dots, \tilde{E}_K)$ with*

$$\tilde{E}_k = \frac{N_0}{|d_k|^2} 2^{\sum_{i=1}^{k-1} R_i} \left(2^{R_k} - 1\right). \tag{1}$$

There exist no other optimum energy tupels if and only if $|d_i| \ne |d_k|, \forall i \ne k$. Single user coding can be applied together with successive cancellation without any loss in performance.

This has already been stated for equal rates in [4, Th. 2], but holds for arbitrarily chosen rates, too. Consider the achievable energy region in the two user case depicted in Fig. 1 (solid line). There is an infinite number of received energy pairs minimizing $\frac{E_1+E_2}{N_0}$ because to increase E_1 implies to decrease E_2 by the same amount. Weighting these energies by $|d_1|^{-2}$ and $|d_2|^{-2}$ with $|d_1|^{-2} \ge |d_2|^{-2}$, respectively, means rescaling the axes of Fig. 1 (see dashed lines). Now, increasing the weigthed energy $\tilde{E}_2$ results in decreasing $\tilde{E}_1$ by a larger amount than $\tilde{E}_2$ is diminished. Therefore, the energy tupel with minimum possible $\tilde{E}_1$ is optimum.

III. Wyners Model

In this section we extend our single cell system to a multi cell system. Then, the received signal Y_m at cell-site m is given as

$$Y_m = \sum_{k=1}^{K} d_{km} \tilde{X}_k + N_m.$$

The $k^{\rm th}$ user's signal received at base station m is attenuated by $d_{km} \in \mathbb{C}$. The power spectral density of the independent complex additive white Gaussian noise N_m, $1 \le m \le M$, is N_0 at all M receivers. Following the assumptions in [2] all signals received at the M cell sites are fed into one joint decoder for the K users. Thus, the rates are defined to be

$$R_k \triangleq I(\tilde{X}_k; Y_1 \dots Y_M), \quad \forall k$$

Defining $\boldsymbol{Y} \triangleq (Y_1, \ldots, Y_M)^T$, $\tilde{\boldsymbol{X}} \triangleq (\tilde{X}_1, \ldots, \tilde{X}_K)^T$, $\boldsymbol{d}_k \triangleq (d_{k1}, \ldots, d_{kM})^T$ and $\boldsymbol{N} \triangleq (N_1, \ldots, N_M)^T$ the complete received signal can be expressed as

$$\boldsymbol{Y} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_K)\,\tilde{\boldsymbol{X}} + \boldsymbol{N} \triangleq \boldsymbol{D}\tilde{\boldsymbol{X}} + \boldsymbol{N}, \qquad (2)$$

with obvious definition of $\boldsymbol{D}$.

We see this channel is equivalent to the direct–sequence code–division multiple–access channel if the k^{th} user's spreading sequence is chosen to be $\boldsymbol{d}_k$, i.e. each spreading sequence consists of M chips. Introducing $\tilde{\boldsymbol{E}} = \operatorname{diag}(\tilde{E}_1, \ldots, \tilde{E}_K)$ the capacity region is given by the conditions (see[1] [5])

$$\sum_{i\in\Omega} R_i \leq \log_2 \det\left(\frac{\boldsymbol{D}_\Omega \tilde{\boldsymbol{E}}_{\Omega,} \boldsymbol{D}_\Omega^H}{N_0} + \boldsymbol{I}_M\right), \quad \forall \Omega \subseteq \{1, \ldots, K\}$$

where the matrix $\tilde{\boldsymbol{E}}_\Omega$ is obtained from $\tilde{\boldsymbol{E}}$ by crossing out all rows *and* columns belonging to users not contained in Ω, $\boldsymbol{D}_\Omega$ results from $\boldsymbol{D}$ by crossing out the columns according to the users being not in Ω. $\boldsymbol{I}_M$ denotes the $M \times M$ identity matrix.

Theorem 2 *For given rate tupel $(R_1, \ldots, R_K), R_k > 0$, the minimum total transmitted energy employing optimal orthogonal transmission is not larger than that using an ideal system if and only if all vectors $\boldsymbol{d}_k$ are collinear with equal length, i.e. $\boldsymbol{d}_k = e^{j\phi_k}\boldsymbol{d}$, $\phi_k \in \mathbb{R}, \forall k$. In this case single user coding combined with successive cancellation is optimum, too.*

Idea of proof: Because of the equivalence of the multi-receiver system described by Eq. (2) with code–division multiple–access we can deduce that the maximum mutual information of each set Ω is minimum if all vectors $\boldsymbol{d}_k$, $k \in \Omega$, are collinear. (This is the worst case because only one dimension is used in an M dimensional space.) Under this condition the smallest upper bound on the sum rate achievable for given transmitted energies E_i, $i \in \Omega$, and fixed $||\boldsymbol{d}_i||^2 \triangleq \boldsymbol{d}_i^H \boldsymbol{d}_i$ is lower bounded by

$$\log_2 \det\left(\frac{\boldsymbol{D}_\Omega \tilde{\boldsymbol{E}}_\Omega \boldsymbol{D}_\Omega^H}{N_0} + \boldsymbol{I}_M\right) \geq \log_2\left(1 + \sum_{i\in\Omega} \frac{\tilde{E}_i}{N_0} ||\boldsymbol{d}_i||^2\right).$$

On the other hand, using an orthogonal transmission scheme the k^{th} user's rate is upper bounded by

$$R_k \leq \Theta_k \log_2\left(1 + \frac{\tilde{E}_{k,\text{orth}} ||\boldsymbol{d}_k||^2}{\Theta_k N_0}\right), \ \forall k,$$

where $\sum_k \Theta_k = 1$. This upper bound on orthogonal multiple–access can be shown to be upper bounded by the lower bound on the unrestricted sum rate.

The attenuation vectors $\boldsymbol{d}_k$ are hardly adjustable by system designers. There is some influence by base station positioning and antenna design onto the amplitude of the elements in $\boldsymbol{D}$, but the phases are almost random with a continuous distribution. Thus, orthogonal transmission is suboptimal with probability 1 due to Theorem 2.

Theorem 3 *The minimum total transmitted energy for a given rate tupel $(R_1, \ldots, R_K), R_k > 0, \forall k$, and given matrix $\boldsymbol{D}$ is achieved by a unique transmitted energy tupel $(\tilde{E}_{1,}, \ldots, \tilde{E}_{K,})$ if $\operatorname{rank}(\boldsymbol{D}) = K$.*

Idea of proof: We have

$$\begin{aligned}\sum_{k\in\Omega} R_k &\leq \log_2 \det\left(\frac{\boldsymbol{D}_\Omega \tilde{\boldsymbol{E}}_\Omega \boldsymbol{D}_\Omega^H}{N_0} + \boldsymbol{I}_M\right), \quad \forall \Omega \subseteq \{1, \ldots, K\} \\ &= \log_2\left(1 + \sum_{\Omega' \subseteq \Omega} \det\left(\boldsymbol{D}_{\Omega'}^H \boldsymbol{D}_{\Omega'}\right) \prod_{i\in\Omega'} \frac{\tilde{E}_i}{N_0}\right). \qquad (3)\end{aligned}$$

The condition $\operatorname{rank}(\boldsymbol{D}) = K$ implies that the terms $\prod_{i=1}^k \tilde{E}_i$ in Eq. (3) do not vanish for all $k \in \Omega$. As $\boldsymbol{D}_\Omega^H \boldsymbol{D}_\Omega$ is positive definite for all Ω, Eq. (3) can be proven to describe a strictly convex set defining a strictly convex constraint for the energy tupels for $|\Omega| > 1$. Moreover, the cost function is convex. Thus, there exists only one unique rate tupel $(\tilde{E}_{1,}, \ldots, \tilde{E}_{K,})_{\text{opt}}$ which minimizes the total transmitted energy.

Note, that the converse of Theorem 3 does not hold. The problem of finding a necessary condition for the existence of a unique solution to the transmitted energy optimization problem remains open.

IV. Conclusions

Considering the standard Gaussian multiple–access channel ($|d_k| = d$, $\forall k$), the number of rate pairs fullfilling the optimality criterion is unlimited. Thus, system designers can choose that realization offering advantages against secondary effects (e.g. Rayleigh fading, channel estimation). In contrast to this, a weighted energy constraint does not provide such freedom in general. In most cases, information theory definitely answers the question, how to distribute the energy among the users. Moreover, if attenuations differ significantly like in cellular mobile radio systems, a severe penalty is expected to be paid for ignoring the fundamental results implied by information theory.

Considering multi–receiver (multi–cell) systems the conditions allowing user separating by orthogonality without increase in required transmitted sum energy are even more restrictive and (with probability 1) not fullfilled in practice. Thus, cell clustering, i.e. frequency reuse partitioning, is far away from being an optimum system approach. In contrast, the same frequency band should be used in all cells in order to keep the cells away from being cells. However, a generalization of these results to dispersive channels is still required to confirm the above conclusions.

References

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[1] In contrast to [5] the factor $1/M$ is not required here, as repetition is not performed by spreading, but by multiple reception.