On the Transport Capacity of a Multiple Access Gaussian Channel

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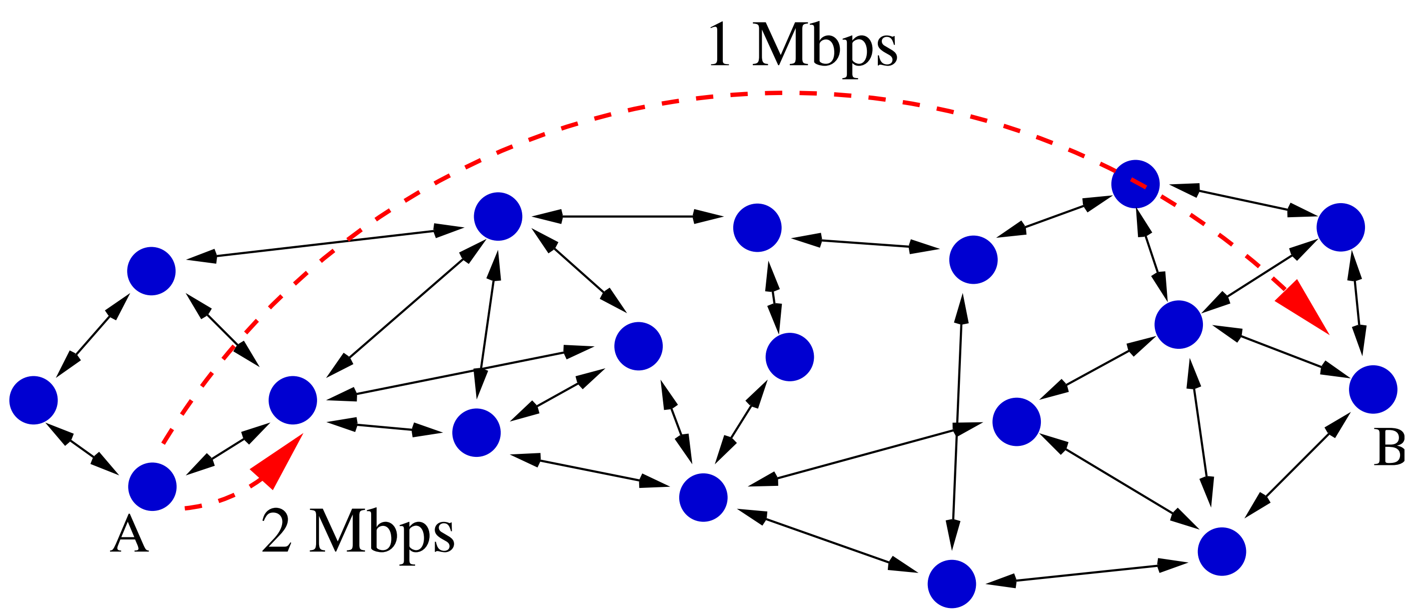
How do we measure the usefulness of a transmission?

- In the following network, node A wants to send information to node B.
- If both transmissions require the same transmitter power and bandwidth, which one is more useful?

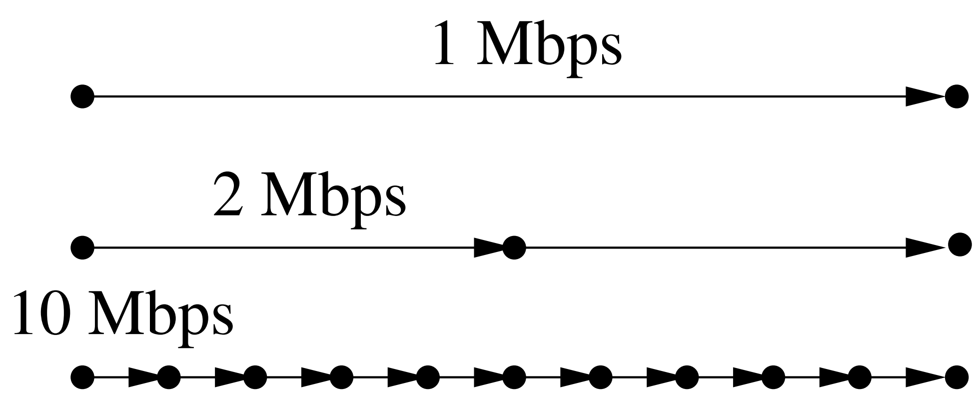


Transport Capacity

- In a *multihop* environment, the appropriate figure of merit for the usefulness of a transmission is the **transport capacity**:

$$\text{Transport Capacity} \triangleq (\text{Covered Distance}) \times (\text{Link Rate}).$$

- Intuition: two links with the same transport capacity are equally useful from the perspective of a network, because the same bandwidth and power is needed to transmit the same volume of traffic.



Previous Work

- P. Gupta and P. R. Kumar, **"The capacity of wireless networks,"** *IEEE Trans. Inform. Theory*, vol. 46, no. 2, pp. 388–404, Mar. 2000.
  - Defined transport capacity and calculated upper and lower bounds in the context of a large wireless ad hoc network.
- A. Reznik and S. Verdú, **"On the transport capacity of a broadcast Gaussian channel,"** *Communications in Information and Systems*, vol. 2, no. 2, pp. 183–216, Dec. 2002.
  - Focus on a broadcast network (one transmitter, many receivers).
  - The authors find the point in the capacity region that maximizes the transport capacity.
- We study a multiple access network (one receiver, many transmitters).
  - We find the point in the capacity region that maximizes the transport capacity.

The reward and decay functions

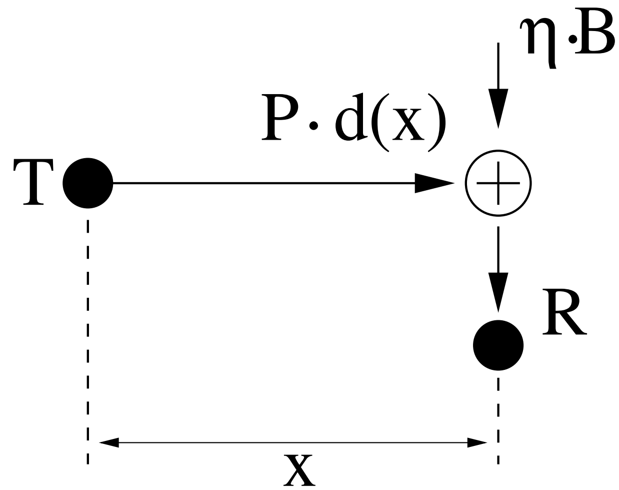
- We will consider a more general notion of transport capacity:

$$\text{Transport Capacity} = (\text{Reward}) \times (\text{Link Rate}).$$

- The successful transmission of a bit along a distance x is rewarded with a **reward function** $r(x)$.
  - $r(0) = 0$.
  - $r(x) \geq r(y)$ when $x \geq y$.
  - Special case: the **monomial reward function** $r(x) = x^\rho$, where ρ is the **reward exponent**.

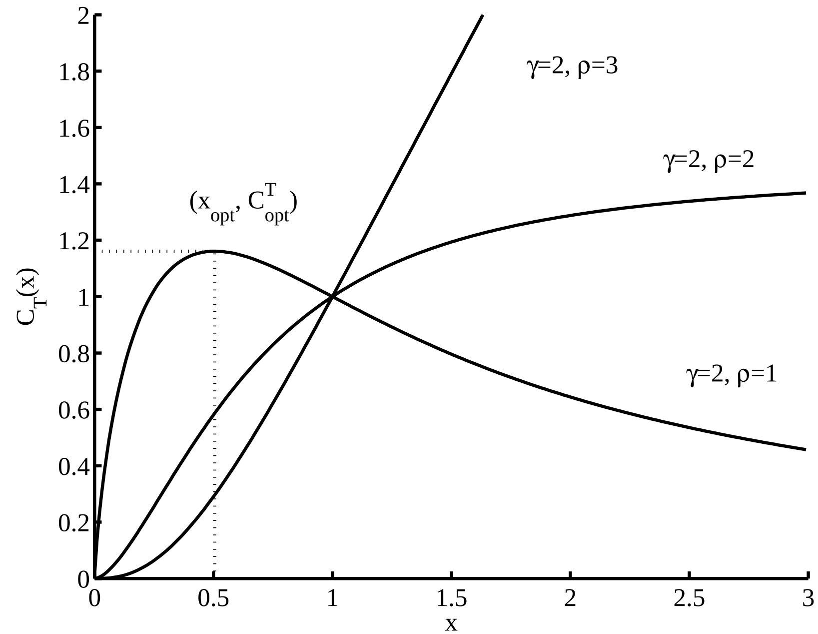
- When a node transmits with power P, the receiver at a distance x receives the signal with power $P \times d(x)$ where $d(x)$ is the **decay function**.
  - Special case: the **monomial decay function** $d(x) = Kx^{-\gamma}$, where $\gamma > 0$ is the **decay exponent**.

Warm Up: the Transport Capacity of a Single Link



- The **transport capacity** $C_T(x) \triangleq r(x) \times B \log_2(1 + \frac{Pd(x)}{\eta B})$.
- For monomial decay and reward functions: $C_T(x) = Bx^\rho \log_2(1 + \frac{KP}{\eta B x^\gamma})$.
- In the special case $\rho = 1$, the transport capacity is the product of the data rate with the link distance.

What is the optimal distance x?



- In the plot, we assume $B = 1$ Hz, $\frac{KP}{\eta B} = 1\ \mathrm{m}^\gamma$.
- $C_T^{\mathrm{opt}} = \sup_{0<x<\infty} Bx^\rho \log_2(1 + \frac{KP}{\eta B x^\gamma})$.
- We are mostly interested in the case $\rho < \gamma$.

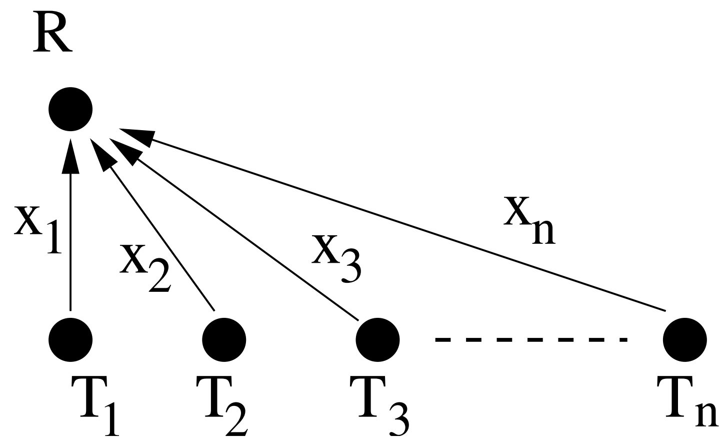
Finding x_{opt}

- By taking the derivative of $C_T(x)$ and using the substitution $y = \log(1 + \frac{A}{x^\gamma}) - \frac{\gamma}{\rho}$, we find that:

$$x_{\text{opt}} = \left[\frac{KP}{\eta B}\right]^{\frac{1}{\gamma}} \left[\frac{1}{e^{g(z_0)} - 1}\right]^{\frac{1}{\gamma}}, \quad C_T^{\text{opt}} = B \left[\frac{KP}{\eta B}\right]^{\frac{\rho}{\gamma}} G(\frac{\rho}{\gamma}).$$

- Where:
  - $G(\frac{\rho}{\gamma}) \triangleq \left[e^{g(z_0)} - 1\right]^{-\frac{\rho}{\gamma}} g(z_0) \log_2(e)$.
  - $z_0 = (-\frac{\gamma}{\rho}) e^{(-\frac{\gamma}{\rho})}$.
  - $g(z) \triangleq W_2(z) - W_1(z)$.
  - $W_2(\cdot)$ and $W_1(\cdot)$ are the two branches of **Lambert's W function**.
  - Lambert's W function is the inverse of the function $y(x) = xe^x$.

The Multiple Access Channel (MAC)



- A single receiver R, multiple transmitters T_i.
- Transmitter T_i is placed at distance x_i from R, and $x_i < x_j$ for $i < j$.
- Transmitter T_i can transmit with maximum power P_i.
- Bandwidth is B, receiver sees AWGN noise of spectral density η.
- Notation: $d_i \triangleq d(x_i)$, $r_i \triangleq r(x_i)$.

The Capacity Region of the MAC

- Each transmitter T_i can send data to the receiver R with rate R_i provided:

$$\sum_{i \in \mathcal{I}} R_i \leq B \log_2(1 + \frac{\sum_{i \in \mathcal{I}} d_i P_i}{\eta B}) \;\; \forall \, \mathcal{I} \subseteq \{1, 2, \ldots, n\}. \tag{1}$$

- Vectors $(R_1, \ldots, R_n)$ that satisfy (1) form the **capacity region**.
  - Therefore, the capacity region is a convex polytope.

- Vertices of the capacity region are achieved by a successive decoding of the n signals:
  - When decoding a signal, only signals that have not yet been decoded interfere. The rest are invisible to the decoder.

Optimal Decoding Order

- With n transmitters, there are $n!$ distinct orders with which their signals could be decoded.

- What is the order of decoding that maximizes the transport capacity?

$$C_T \triangleq \sum_{i=1}^{n} r_i R_i.$$

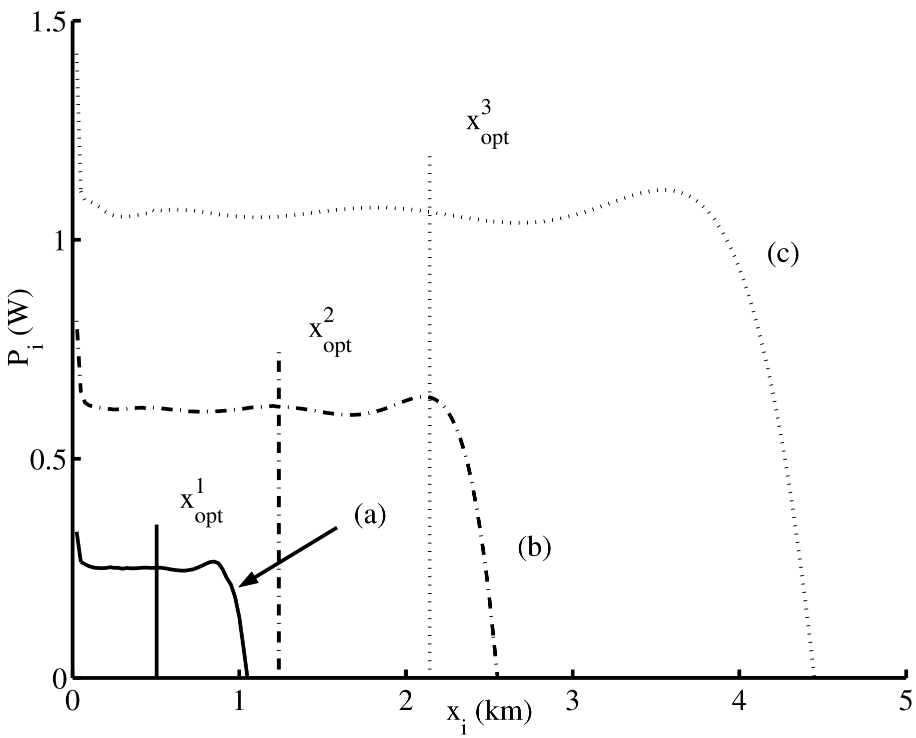
- **Theorem:** As long as the reward function is monotonically non-decreasing, we should decode signals starting from the near nodes and moving outwards [1].

[1] Similar result has already appeared in a different context: S. Vishwanath, S. A. Jafar, and A. J. Goldsmith, "Optimum power and rate allocation strategies for multiple access fading channels," in *Proc. Spring IEEE VTC*, vol. 4, Rhodes, Greece, May 2001, pp. 2888–2892.

The MAC with a sum-power constraint

- Until now, we assumed that each node has its own power constraint: $P_i \leq P_i^{\max}$.
- We now assume that nodes have a *sum-power constraint*: $\sum_{i=1}^{n} P_i \leq P_0$.
- **What is the optimal power distribution?**
- Motivation:
  - When designing the network, we want to know how to distribute battery power among nodes.
  - We may be required to satisfy a bound on the interference experienced by distant networks.
  - We can actually show this problem is **convex**.

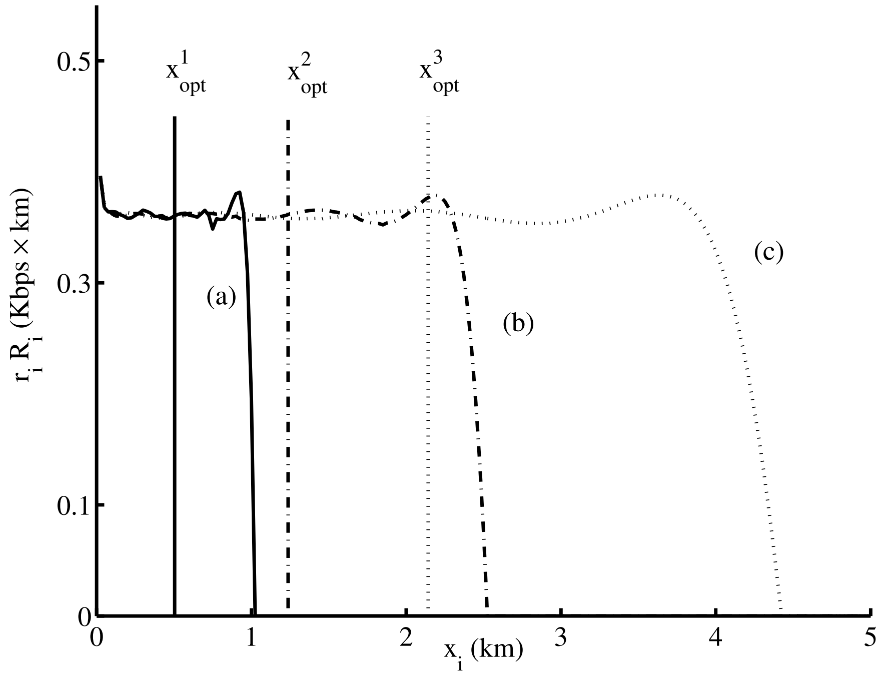
A Numerical Example: Distribution of Power



- The receiver is placed at the origin.
- 200 transmitters, that are placed uniformly every 25 m. (Closest is placed 25 m from the receiver.)
- $B = 10$ MHz, $\eta = 10^{-16}\ \frac{\mathrm{W}}{\mathrm{Hz}}$, $\rho = 1$, $\gamma = 2$, $K = 10^{-4}\ \mathrm{m}^2$.
- Three different powers:

  (a) $P_0^1 = 10\ W$.
  (b) $P_0^2 = 60\ W$.
  (c) $P_0^3 = 180\ W$.

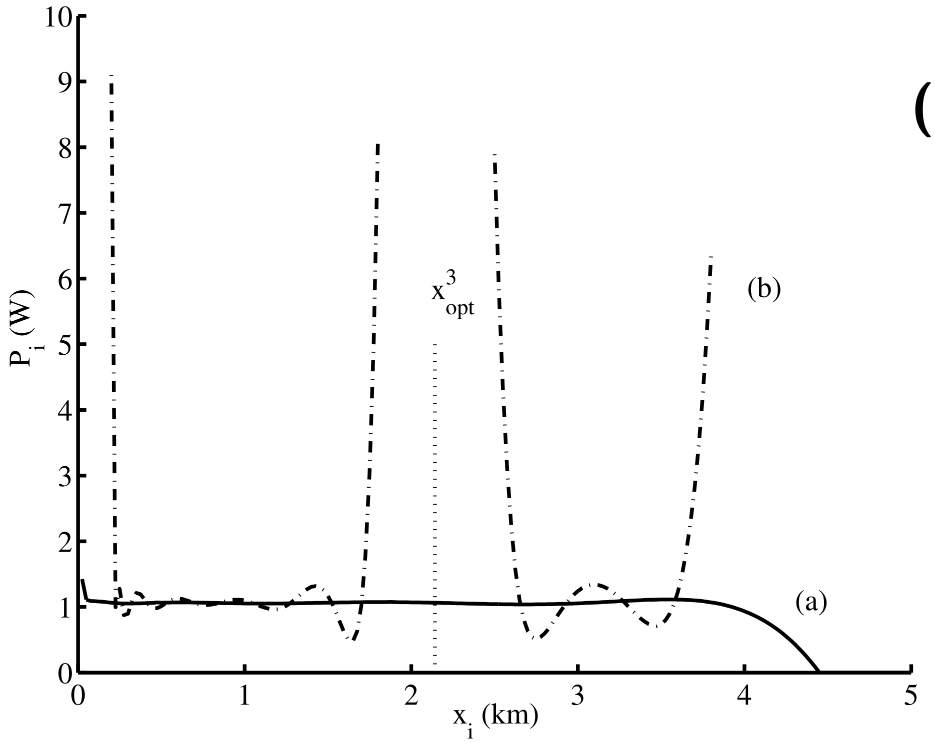
Transport Capacity Distribution



- **Conjecture:** As the transmitters are packed closer, the optimal power distribution and the corresponding transport capacity distribution converge to step functions of an appropriate height and width.

- In this example, the transport capacity increases by 15% with respect to the single-transmitter case.

Introducing "empty regions"



(a) 200 nodes, placed uniformly, every $25\ \mathrm{m}$, in the interval $[25\ \mathrm{m}, 5000\ \mathrm{m}]$.

(b) 118 nodes, placed uniformly, every $25\ \mathrm{m}$, in the intervals $[200\ \mathrm{m}, 1800\ \mathrm{m}]$ and $[2500\ \mathrm{m}, 3800\ \mathrm{m}]$.

- Nodes close to the "empty regions" transmit with higher power.
- Transport capacity is reduced by only 2%, therefore system is very robust!
- Similar behavior observed in the case of the broadcast channel (Reznik & Verdu).

Conclusions

- We study transport capacity, a largely unexplored area.
- We present a closed form expression of the optimum distance between a single transmitter and a single receiver, and the corresponding transport capacity.
- In the multiple access channel, the decoding order that maximizes the transport capacity is remarkably simple.
  - Signals from nodes close to the receiver are decoded first.
- In the multiple access channel, under a sum-power constraint, the optimum power allocation may be found by solving a convex optimization problem.
  - In the optimum power allocation, nodes up to a certain distance divide the available power equally among themselves.