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PERFORMANCE ANALYSIS OF OPTIMAL CHANNEL ESTIMATION AND MULTIUSER DETECTION IN A RANDOMLY-SPREAD CDMA CHANNEL*

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Abstract This paper analyzes performance of optimal channel estimation and multiuser detection (MUD) in a block-fading code-division multiple-access (CDMA) channel on the assumptions of random spreading and large-system limit, by using the replica method developed in statistical mechanics. The authors find that the asymptotic spectral efficiency of the linear minimum mean-squared error (LMMSE) MUD which was proposed and analyzed by Evans and Tse in 2000 is indistinguishable from that of the optimal MUD for small system loads. Our results imply that performance of MUD scarcely improves even if one spends more computational cost than that of the LMMSE MUD, i.e., at most the cube of the number of users, on the above-described conditions.

Key words Channel estimation, code-division multiple-access (CDMA) systems, multiuser detection, replica method, spectral efficiency.

1 Introduction

Code-division multiple-access (CDMA) schemes are used in third generation cellular networks. CDMA uplink allows multiple users within a cell to simultaneously communicate with a base station in the same frequency band by assigning different spreading sequences to different users. A spreading sequence in practical CDMA systems is commonly generated by combining a short orthogonal spreading sequence over one symbol period and a long pseudorandom sequence over multiple symbol periods. As a result, the base station receives a superposition of all

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users' signals, which is decomposed into three components: signal, multiple-access interference (MAI), and noise. Conventional CDMA systems regard the MAI as an additive white Gaussian noise (AWGN), and uses the single-user matched filter (SUMF)[1] for the detection of each data symbol. Explicitly utilizing statistical properties of MAI improves the detection of each data symbol at the base station. This issue is called multiuser detection (MUD)[1] and is the main topic of this paper.

MUD realizing near-optimal performance with low computational complexity is desired. It was shown that the jointly-optimal (JO) MUD, which minimizes the block error probability of data symbols, is nondeterministic polynomial-time hard (NP-hard) in the number of users K[2]. The individually-optimal (IO) MUD, which minimizes the bit error rate of each data symbol, also has extremely high complexity[1]: No polynomial-time algorithm of the IO MUD is known. The computational cost of the minimum mean-squared error (MMSE) MUD is not less than that of the IO MUD because the hard decision of an MMSE estimate equals the decision of the IO MUD. The MMSE multiuser detector provides the best performance among them in terms of information-theoretical capacity. In order to reduce the time-complexity of MUD, some sub-optimal multiuser detectors with at most $O(K^3)$ time-complexity were proposed, such as the decorrelator (DEC) and the linear MMSE (LMMSE) multiuser detector. The LMMSE multiuser detector provides better performance than the SUMF and the DEC multiuser detector. It is a fundamentally important issue in MUD to answer whether or not one acquires some improvement of performance by spending more computational cost than that of the LMMSE multiuser detector, although this issue is generally difficult.

An efficient approach to circumventing this difficulty is to assume random spreading and to consider large-system limit, in which K and the spreading factor N, which is defined as the ratio of chip rate to symbol rate, go to infinity while the system load $\beta = K/N$ is kept constant. The performance of the LMMSE multiuser detector in an unfaded CDMA channel was analyzed on these assumptions[3−4]. On the other hand, the performances of the optimal multiuser detectors in the same channel were evaluated by using the replica method[5], which is a powerful approach developed in statistical mechanics. Furthermore, it was shown that the performance gap between the MMSE and LMMSE multiuser detectors in a fading CDMA channel with perfect channel state information (CSI) at the base station, which means that the spreading sequences and the fading process are perfectly known to the base station, is negligible for small β[6]. These previous studies imply that one can realize near-optimal MUD with polynomial time-complexity in K and N in randomly-spread CDMA channels if K and N are sufficiently large and $\beta \ll 1$.

The goal of this paper is to extend the previous results to the case in which the fading process is not perfectly known to the base station. In order to simplify the performance evaluation of MUD in this case, we consider a block-fading channel model, in which channel gains do not change during a coherent interval and are independently drawn from a distribution at the beginning of the next coherent interval[7]. Evans and Tse[8] considered channel estimation and MUD in a block-fading CDMA channel, and proposed and analyzed an LMMSE multiuser detector. Their LMMSE multiuser detector utilizes soft information about channel gains, which consists of estimates of the channel gains and statistical properties of their estimation errors, for estimating data symbols. In this paper, we evaluate the performance of an optimal multiuser detector in a block-fading CDMA channel in order to assess the performance gap between the two multiuser detectors.

Some results presented in this paper are based on the replica method. It is useful for the analysis of nonlinear multiuser detectors[5−6,9−13], and successfully reproduces some known results for linear multiuser detectors although it lacks mathematical justification. Therefore, we call results derived by using the replica method not "theorems" but "claims".

Throughout this paper, logarithms are taken to base 2. $\mathcal{CN}(0,\sigma^2)$ denotes a circularly symmetric complex Gaussian distribution with variance σ^2, whose probability density function (pdf) $p(z)$ is given by

$$p(z) = \frac{1}{\pi\sigma^2}\exp\Big(-\frac{|z|^2}{\sigma^2}\Big), \quad z \in \mathbb{C}. \tag{1}$$

This paper is organized as follows. We define a block-fading CDMA channel and derive an MMSE channel estimator and an MMSE multiuser detector in Section 2. The asymptotic performance of the MMSE multiuser detector is analyzed in Section 3. We numerically compare it with that of the LMMSE multiuser detector[8] in Section 4. Section 5 concludes this paper. We describe a crucial calculation required to derive the main result in Appendix.

2 Channel Model

2.1 *K*-User CDMA Channel

Practical CDMA uplink is asynchronous because synchronization among users is not controlled. For simplicity of analysis, we consider a perfectly-synchronous K-user CDMA uplink over a block-fading channel with a coherent time T_{c}. User k transmits the product of an input $x_{k,t} \in \mathbb{C}$ and a spreading sequence $\{s_{n,k,t} \in \mathbb{C} : n = 1, 2, \cdots, N\}$, in which N corresponds to the spreading factor, in the tth symbol period within a coherent interval. The nth chip-sampled received output $y_{n,t}$ in the tth symbol period is given by

$$y_{n,t} = \frac{1}{\sqrt{N}}\sum_{k=1}^{K} h_k s_{n,k,t} x_{k,t} + w_{n,t}, \quad n = 1, 2, \cdots, N, \ \ t = 1, 2, \cdots, T_{\mathrm{c}}, \tag{2}$$

where $h_k \in \mathbb{C}$ denotes the channel gain between the kth user and the base station and does not change in the coherent interval, and where $w_{n,t} \sim \mathcal{CN}(0, N_0)$ represents an AWGN.

We assume that the base station has CSI except realizations of the channel gains $\mathcal{H} = \{h_k : k = 1, 2, \cdots, K\}$. It knows all spreading sequences and the statistical properties of $\{h_k\}$, $\{x_{k,t}\}$, and $\{w_{n,t}\}$ for all k, t, n. The base station consists of a channel estimator and a multiuser detector followed by per-user decoders. In this paper, we consider only MMSE channel estimation and MMSE MUD. The first τ symbol periods are assigned to a training phase in order for an MMSE channel estimator to estimate the channel gains $\mathcal{H}$, and the remaining $\tau' = T_{\mathrm{c}} - \tau$ symbol periods are assigned to a communication phase, in which an MMSE multiuser detector estimates data symbols. In the training phase, user k transmits pilot symbols $\{p_{k,t}\}$, i.e., $x_{k,t} = p_{k,t}$ for $t = 1, 2, \cdots, \tau$. The MMSE channel estimator estimates the channel gains $\mathcal{H}$ on the basis of the knowledge $\mathcal{I} = \{\mathcal{P},\ \mathcal{S}_{\mathrm{tr}},\ \mathcal{Y}_{\mathrm{tr}}\}$ of the pilot symbols $\mathcal{P} = \{p_{k,t} : k = 1, 2, \cdots, K,\ t = 1, 2, \cdots, \tau\}$, the spreading sequences $\mathcal{S}_{\mathrm{tr}} = \{s_{n,k,t} : n = 1, 2, \cdots, N,\ k = 1, 2, \cdots, K,\ t = 1, 2, \cdots, \tau\}$, and the received outputs $\mathcal{Y}_{\mathrm{tr}} = \{y_{n,t} : n = 1, 2, \cdots, N,\ t = 1, 2, \cdots, \tau\}$ in the training phase, and provides soft information about the channel gains $\mathcal{H}$ to the MMSE multiuser detector. In the communication phase, user k transmits coded data symbols $\{b_{k,t}\}$, i.e., $x_{k,t} = b_{k,t}$ for $t = \tau+1, \tau+2, \cdots, T_{\mathrm{c}}$. The MMSE multiuser detector estimates the data symbols $\mathcal{B}_t = \{b_{k,t} : k = 1, 2, \cdots, K\}$ from the spreading sequences $\mathcal{S}_t = \{s_{n,k,t} : n = 1, 2, \cdots, N,\ k = 1, 2, \cdots, K\}$ and the received outputs $\mathcal{Y}_t = \{y_{n,t} : n = 1, 2, \cdots, N\}$ in the same symbol period t, along with the soft information about the channel gains $\mathcal{H}$ provided by the channel estimator, and feeds soft information about each data symbol to the corresponding decoder.

We consider this type of MUD in order to make a fair comparison of the LMMSE MUD proposed by Evans and Tse[8] and the MMSE MUD. We remark that this MUD is sub-optimal because any estimator $T(\mathcal{Y}_t, \mathcal{S}_t, \mathcal{I})$ of $\mathcal{B}_t$ depending only on $\mathcal{S}_t$, $\mathcal{Y}_t$, and $\mathcal{I}$ is not sufficient. Intuitively, one can understand this fact as follows: The estimation of the channel gains $\mathcal{H}$ improves by using soft information about $\mathcal{B}_{t'}$, $t' \neq t$, as effective pilot symbols, and the improved estimates of the channel gains $\mathcal{H}$ can be utilized for the MUD of $\mathcal{B}_t$. Consequently, the estimation of $\mathcal{B}_t$ is refined by using the estimates of $\mathcal{B}_{t'}$, which do depend on $\mathcal{S}_{t'}$ and $\mathcal{Y}_{t'}$. More generally, the estimation of $\mathcal{B}_t$ improves by utilizing the estimates of the other data symbols $\{\mathcal{B}_{t'} : \text{for all } t', \text{ except } t' = t\}$, which depend on $\{\mathcal{Y}_{t'},\ \mathcal{S}_{t'} : \text{for all } t', \text{ except } t' = t\}$, as effective pilot symbols. This discussion is consistent with the fact that a sufficient statistic of $\mathcal{B}_t$ is given by the form $T(\{\mathcal{Y}_t : \text{for all } t\}, \{\mathcal{S}_t : \text{for all } t\}, \mathcal{I})$.

2.2 MMSE Channel Estimation

Let us define soft information about the channel gains $\mathcal{H}$ provided to the MMSE multiuser detector. The MMSE channel estimator, which minimizes the mean-square error (MSE) of channel estimates conditioned on $\mathcal{P}$ and $\mathcal{S}_{\mathrm{tr}}$, coincides with the posterior mean estimator (PME) of the channel gains given $\mathcal{I}$. Therefore, the soft information is given as the posterior pdf of $\mathcal{H}$ given $\mathcal{I}$:

$$p(\mathcal{H}|\mathcal{I} = \{\mathcal{P},\ \mathcal{S}_{\mathrm{tr}},\ \mathcal{Y}_{\mathrm{tr}}\}) = \frac{p(\mathcal{Y}_{\mathrm{tr}}|\mathcal{H}, \mathcal{S}_{\mathrm{tr}}, \mathcal{P})p(\mathcal{H})}{\int p(\mathcal{Y}_{\mathrm{tr}}|\mathcal{H}, \mathcal{S}_{\mathrm{tr}}, \mathcal{P})p(\mathcal{H})d\mathcal{H}}, \tag{3}$$

where $p(\mathcal{H})$ is the pdf of $\mathcal{H}$, and where the conditional pdf $p(\mathcal{Y}_{\mathrm{tr}}|\mathcal{H}, \mathcal{S}_{\mathrm{tr}}, \mathcal{P})$ characterizes the CDMA channel (2) in the training phase.

2.3 MMSE Multi-User Detection

The MMSE multiuser detector estimates the data symbols $\mathcal{B}_t$ from $\mathcal{S}_t$, $\mathcal{Y}_t$, and $p(\mathcal{H}|\mathcal{I})$, and provides soft information about each data symbol to the corresponding decoder. The soft information provided to the kth user's decoder is given as the marginal posterior pdf of $b_{k,t}$ given $\mathcal{S}_t$, $\mathcal{Y}_t$, and $p(\mathcal{H}|\mathcal{I})$:

$$p(b_{k,t}|\mathcal{S}_t, \mathcal{Y}_t, \mathcal{I}) = \frac{\int p(\mathcal{Y}_t|\mathcal{S}_t, \mathcal{B}_t, \mathcal{I})p(\mathcal{B}_t)d\mathcal{B}_{\backslash k,t}}{\int p(\mathcal{Y}_t|\mathcal{S}_t, \mathcal{B}_t, \mathcal{I})p(\mathcal{B}_t)d\mathcal{B}_t}, \tag{4}$$

where $p(\mathcal{B}_t)$ is the pdf of $\mathcal{B}_t$, $\mathcal{B}_{\backslash k,t}$ denotes $\mathcal{B}_{\backslash k,t} = \{b_{k',t} : \text{for all } k', \text{ except } k' = k\}$, and $p(\mathcal{Y}_t|\mathcal{S}_t, \mathcal{B}_t, \mathcal{I})$ is given by

$$p(\mathcal{Y}_t|\mathcal{S}_t, \mathcal{B}_t, \mathcal{I}) = \int p(\mathcal{Y}_t|\mathcal{H}, \mathcal{S}_t, \mathcal{B}_t)p(\mathcal{H}|\mathcal{I})d\mathcal{H}. \tag{5}$$

In Equation (5), $p(\mathcal{Y}_t|\mathcal{H}, \mathcal{S}_t, \mathcal{B}_t)$ represents the CDMA channel (2) in the $t\,(> \tau)$th symbol period. The computational cost of the MMSE MUD for practical data modulation is exponential in K if one directly calculates the marginalizations in (4).

The performance of the MMSE multiuser detector is measured with spectral efficiency[7]. Let $\widehat{b}_{k,t} \in \mathbb{C}$ denote a random variable following the marginal posterior pdf (4). The spectral efficiency of the MMSE multiuser detector is defined as

$$C = \frac{1}{LT_{\mathrm{c}}} \sum_{k=1}^{K} \sum_{t=\tau+1}^{T_{\mathrm{c}}} I(b_{k,t}; \widehat{b}_{k,t}|\mathcal{S}_t, \mathcal{I}), \tag{6}$$

where $I(b_{k,t}; \widehat{b}_{k,t}|\mathcal{S}_t, \mathcal{I})$ denotes the mutual information between $b_{k,t}$ and $\widehat{b}_{k,t}$ conditioned on $\mathcal{S}_t$ and $\mathcal{I}$:

$$I(b_{k,t}; \widehat{b}_{k,t}|\mathcal{S}_t, \mathcal{I}) = \int p(\widehat{b}_{k,t}|b_{k,t}, \mathcal{S}_t, \mathcal{I})p(b_{k,t}) \log \frac{p(\widehat{b}_{k,t}|b_{k,t}, \mathcal{S}_t, \mathcal{I})}{\int p(\widehat{b}_{k,t}|b_{k,t}, \mathcal{S}_t, \mathcal{I})p(b_{k,t})db_{k,t}} db_{k,t} d\widehat{b}_{k,t} \tag{7}$$

with

$$p(\widehat{b}_{k,t}|b_{k,t},\mathcal{S}_t,\mathcal{I}) = \int p(b_{k,t}=\widehat{b}_{k,t}|\mathcal{S}_t,\mathcal{Y}_t,\mathcal{I})p(\mathcal{Y}_t|\mathcal{S}_t,\mathcal{B}_t,\mathcal{I})p(\mathcal{B}_{\backslash k,t})d\mathcal{Y}_t d\mathcal{B}_{\backslash k,t}. \quad (8)$$

The conditional pdf (8) characterizes the equivalent channel between the kth user and the corresponding decoder.

Let us consider a case in which the rates of all users are equal, in order to discuss the operational meaning of the spectral efficiency (6). It is possible for each user to realize reliable communication if and only if the sum rate is less than C[14]. Therefore, the spectral efficiency (6) provides an information-theoretical performance limitation for the case in which the MMSE MUD followed by per-user decoders is used.

2.4 Assumptions

Throughout this paper, we consider quadrature phase-shift keying (QPSK) modulation in the training and communication phases, i.e., we assume that all transmitted symbols $\{x_{k,t}$: for all k and $t\}$ are independent and identically distributed (i.i.d.), and that the pdf of $x_{k,t}$ is given by

$$p(x_{k,t}) = \frac{1}{4}\sum_{j=\{-1,1\}}\sum_{k=\{-1,1\}}\delta\left(x_{k,t}-\sqrt{P}\frac{(j+k\mathrm{i})}{\sqrt{2}}\right), \quad (9)$$

where i denotes the imaginary unit. We assume random spreading†: $\{\mathrm{Re}[s_{n,k,t}]$, $\mathrm{Im}[s_{n,k,t}]$: for all n, k, and $t\}$ are assumed to be i.i.d. zero-mean random variables with variance $1/2$. Furthermore, we assume that $\{h_k : k=1,2,\cdots,K\}$ are i.i.d. circularly symmetric complex Gaussian random variables with unit variance. The last assumption is called the assumption of i.i.d. Rayleigh fading. Therefore, the MMSE channel estimator is equivalent to the LMMSE channel estimator[8].

3 Main Results

3.1 Summary of Results

We focus on the conditional pdf (8) in order to analytically evaluate the mutual information (7). It is difficult to calculate Equation (8) except for small K and N, because its evaluation includes the marginalization with respect to $\mathcal{B}_{\backslash k,t}$. We consider the large-system limit, in which K and N go to infinity with $\beta = K/N$ kept constant, in order to circumvent this difficulty. $p(\widehat{b}_{k,t}|b_{k,t},\mathcal{S}_t,\mathcal{I})$ depends on realizations of $\mathcal{S}_t$ and $\mathcal{I}$ for finite K and N. However, it was shown that it converges to a conditional pdf which does not depend on any realizations of $\mathcal{S}_t$, in the large-system limit if the LMMSE multiuser detector is considered[8]. Let us assume that this self-averaging property of Equation (8) with respect to $\mathcal{S}_t$ holds even for the MMSE multiuser detector.

Assumption 1 (Self-averaging property) The conditional pdf (8) converges in law to an asymptotic conditional pdf which is independent of $\mathcal{S}_t$, in the large-system limit.

This kind of self-averaging property was also assumed in previous studies[6,13]. Assumption 1 is assumed because it is beyond the scope of this paper to prove the self-averaging property.

Our main result is the decoupling of the randomly-spread CDMA channel (2) in the large-system limit: It is asymptotically decomposed into a bank of single-user fading channels, and the mutual information (7) coincides with the mutual information between a data symbol and its estimate in the single-user fading channel. Therefore, the spectral efficiency (6) is analytically

†Note that $\{\mathrm{Re}[s_{n,k,t}]$, $\mathrm{Im}[s_{n,k,t}]$: for all $n\}$ are not identically distributed for practical spreading sequences.

evaluated via the spectral efficiency of a bank of the single-user fading channels. We first define the single-user fading channels and then provide a precise statement as a claim.

Definition 1 (Single-user fading channel) We define a single-user fading channel for the kth user as

$$z_{k,t} = h_k x_{k,t} + v_{k,t}, \quad v_{k,t} \sim \mathcal{CN}(0, \sigma_t^2), \quad \text{for } t = 1, 2, \cdots, T_c. \tag{10}$$

The receiver consists of an MMSE channel estimator, an MMSE detector, and the kth user's decoder. Let us define the posterior pdf of h_k given the knowledge $\mathcal{I}_k = \{\mathcal{Z}_k,\ \mathcal{P}_k\}$ of the received outputs $\mathcal{Z}_k = \{z_{k,t} : t = 1, 2, \cdots, \tau\}$ and the pilot symbols $\mathcal{P}_k = \{p_{k,t} : t = 1, 2, \cdots, \tau\}$ of the kth user as

$$p(h_k|\mathcal{I}_k) = \frac{p(\mathcal{Z}_k|h_k, \mathcal{P}_k)p(h_k)}{\int p(\mathcal{Z}_k|h_k, \mathcal{P}_k)p(h_k)dh_k}, \tag{11}$$

where $p(\mathcal{Z}_k|h_k, \mathcal{P}_k)$ represents the single-user fading channel (10) in the training phase. The MMSE channel estimator constructs the posterior pdf (11) in the training phase and provides it to the MMSE detector. In the communication phase, the MMSE detector estimates each data symbol $b_{k,t}$ from $z_{k,t}$ and $p(h_k|\mathcal{I}_k)$, and provides soft information about the data symbol $b_{k,t}$ to the kth user's decoder in the form of the posterior pdf of $b_{k,t}$ given $z_{k,t}$ and $\mathcal{I}_k$:

$$p(b_{k,t}|z_{k,t}, \mathcal{I}_k) = \frac{p(z_{k,t}|b_{k,t}, \mathcal{I}_k)p(b_{k,t})}{\int p(z_{k,t}|b_{k,t}, \mathcal{I}_k)p(b_{k,t})db_{k,t}} \tag{12}$$

with

$$p(z_{k,t}|b_{k,t}, \mathcal{I}_k) = \int p(z_{k,t}|h_k, b_{k,t})p(h_k|\mathcal{I}_k)dh_k, \tag{13}$$

where $p(z_{k,t}|h_k, b_{k,t})$ characterizes the single-user fading channel (10) in the $t\,(> \tau)$th symbol period.

Let $\tilde{b}_{k,t} \in \mathbb{C}$ denote a random variable following the posterior pdf (12). The spectral efficiency of this MMSE detector is given by

$$C_k = \frac{1}{T_c} \sum_{t=\tau+1}^{T_c} I(b_{k,t}; \tilde{b}_{k,t}|\mathcal{I}_k), \tag{14}$$

where the mutual information $I(b_{k,t}; \tilde{b}_{k,t}|\mathcal{I}_k)$ between $b_{k,t}$ and $\tilde{b}_{k,t}$ conditioned on $\mathcal{I}_k$ is defined as

$$I(b_{k,t}; \tilde{b}_{k,t}|\mathcal{I}_k) = \int p(\tilde{b}_{k,t}|b_{k,t}, \mathcal{I}_k)p(b_{k,t}) \log \frac{p(\tilde{b}_{k,t}|b_{k,t}, \mathcal{I}_k)}{\int p(\tilde{b}_{k,t}|b_{k,t}, \mathcal{I}_k)p(b_{k,t})db_{k,t}} db_{k,t}d\tilde{b}_{k,t} \tag{15}$$

with

$$p(\tilde{b}_{k,t}|b_{k,t}, \mathcal{I}_k) = \int p(b_{k,t} = \tilde{b}_{k,t}|z_{k,t}, \mathcal{I}_k)p(z_{k,t}|b_{k,t}, \mathcal{I}_k)dz_{k,t}. \tag{16}$$

The spectral efficiency (6) coincides with the spectral efficiency (14) by letting the variances $\{\sigma_t^2\}$ be particular values.

Claim 1 *The spectral efficiency* (6) *of the MMSE multiuser detector converges in probability to the expectation of the spectral efficiency* (14) *of the MMSE detector in the large-system limit:*

$$\lim_{K=\beta N \to \infty} C = \mathbb{E}_{\mathcal{I}_k}[C_k] \quad \textit{in probability.} \tag{17}$$

In evaluating the right-hand side of Equation (17)*, one should let the variances* $\{\sigma_t^2\}$ *be* $\sigma_{\rm tr}^2$ *for* $t = 1, 2, \cdots, \tau$*, and be* $\sigma_{\rm c}^2$ *for* $t = \tau+1, \tau+2, \cdots, T_{\rm c}$*.* $\sigma_{\rm tr}^2$ *is the unique solution of the following fixed-point equation:*

$$\sigma_{\rm tr}^2 = N_0 + \beta P \xi^2, \tag{18}$$

where ξ^2 *is the asymptotic MSE of the MMSE channel estimation given by*

$$\xi^2 = \frac{\sigma_{\rm tr}^2}{\tau P + \sigma_{\rm tr}^2}. \tag{19}$$

On the other hand, $\sigma_{\rm c}^2$ *satisfies the fixed-point equation:*

$$\sigma_{\rm c}^2 = N_0 + \beta \frac{P\xi^2 \sigma_{\rm c}^2}{P\xi^2 + \sigma_{\rm c}^2} + \beta \left(\frac{\sigma_{\rm c}^2}{P\xi^2 + \sigma_{\rm c}^2} \right)^2 \mathbb{E}\left[|\langle h_k; \mathcal{I}_k \rangle_{\rm tr} (b_{k,t} - \langle b_{k,t}; z_{k,t}, \mathcal{I}_k \rangle_{\rm c})|^2 \right], \tag{20}$$

where the posterior mean $\langle h_k; \mathcal{I}_k \rangle_{\rm tr}$ *of* h_k *conditioned on* $\mathcal{I}_k$ *is given by*

$$\langle h_k; \mathcal{I}_k \rangle_{\rm tr} = \int h_k p(h_k | \mathcal{I}_k) dh_k, \tag{21}$$

and where the posterior mean $\langle b_{k,t}; z_{k,t}, \mathcal{I}_k \rangle_{\rm c}$ *of* $b_{k,t}$ *given* $z_{k,t}$ *and* $\mathcal{I}_k$ *is defined as*

$$\langle b_{k,t}; z_{k,t}, \mathcal{I}_k \rangle_{\rm c} = \int b_{k,t} p(b_k | z_{k,t}, \mathcal{I}_k) dh_k db_{k,t}. \tag{22}$$

The fixed-point equation (20) *can have multiple solutions. In this case, one should select the solution to minimize the following quantity*

$$\beta \mathbb{E}_{\mathcal{I}_k} \left[I(b_{k,t}; \widetilde{b}_{k,t} | \mathcal{I}_k) \right] + {\rm D}(\mathcal{CN}(0, N_0) \| \mathcal{CN}(0, \sigma_{\rm c}^2)), \tag{23}$$

where ${\rm D}(p\|q)$ *denotes the Kullback-Leibler divergence with the logarithm to base* 2 *between pdfs* $p(x)$ *and* $q(x)$*:*

$${\rm D}(p\|q) = \int p(x) \log \frac{p(x)}{q(x)} dx. \tag{24}$$

We note that the second term of the right-hand side of Equation (18) depends on $\sigma_{\rm tr}^2$ through ξ^2, and that the fixed-point equation (18) was originally derived by Evans and Tse[8]. Claim 1 implies that the equivalent channel between the kth user and the corresponding decoder in the large-system limit looks like the single-user fading channel (10) with the effective MMSE channel estimator and MMSE detector. This result is an extension of the decoupling principle[6] of randomly-spread CDMA channels.

The expression of the fixed-point equation (20) implies that MAI is represented by two effects: The second term of the right-hand side of Equation (20) corresponds to contribution from an effective unfaded AWGN channel, and the third term corresponds to interference from the effective single-user fading channel with perfect CSI at the receiver

$$\widetilde{z}_{k,t} = \langle h_k; \mathcal{I}_k \rangle_{\rm tr} b_{k,t} + \widetilde{v}_{k,t}, \quad \widetilde{v}_{k,t} \sim \mathcal{CN}(0, P\xi^2 + \sigma_{\rm c}^2). \tag{25}$$

The asymptotic MSE ξ^2 vanishes like $\xi^2 = O(\tau^{-1})$ as the length τ of the pilot sequences is sufficiently large. In this case, the second term of Equation (20) also vanishes and the fixed-point equation (20) is reduced to that for perfect CSI[6].

Equations (14) and (20) can be easily evaluated with some numerical integration because the posterior mean $\langle h_k; \mathcal{I}_k \rangle_{\text{tr}}$ follows a circularly symmetric complex Gaussian random variable with variance $1 - \xi^2$.

3.2 Review of LMMSE

We review the performance analysis of the LMMSE multiuser detector proposed by Evans and Tse[8], in order to compare it with our MMSE multiuser detector. The definition of the LMMSE multiuser detector is omitted because it is described in the reference[8] in detail. The time-complexity of the LMMSE MUD is at most $O(K^3)$ in the large-system limit.

Theorem 1[8] *The spectral efficiency C_{LMMSE} of the LMMSE multiuser detector converges in probability to*

$$\lim_{K=\beta N\to\infty} C_{\text{LMMSE}} = \frac{1}{T_{\text{c}}} \sum_{t=\tau+1}^{T_{\text{c}}} \mathbb{E}_{\mathcal{I}_k}\left[I(b_{k,t}; z_{k,t} | \mathcal{I}_k) \right] \quad \text{in probability}, \tag{26}$$

in the large-system limit, in which

$$I(b_{k,t}; z_{k,t}|\mathcal{I}_k) = \int p(z_{k,t}|b_{k,t}, \mathcal{I}_k) p(b_{k,t}) \log \frac{p(z_{k,t}|b_{k,t}, \mathcal{I}_k)}{\int p(z_{k,t}|b_{k,t}, \mathcal{I}_k) p(b_{k,t}) db_{k,t}} db_{k,t} dz_{k,t}. \tag{27}$$

The variances σ_t^2 for $t = \tau+1, \tau+2, \cdots, T_{\text{c}}$, which are included in $p(z_{k,t}|b_{k,t}, \mathcal{I}_k)$, are given by σ_{L}^2, which is the unique solution of the following fixed-point equation:

$$\sigma_{\text{L}}^2 = N_0 + \beta \frac{P\xi^2\sigma_{\text{L}}^2}{P\xi^2 + \sigma_{\text{L}}^2} + \beta \left(\frac{\sigma_{\text{L}}^2}{P\xi^2 + \sigma_{\text{L}}^2} \right)^2 \mathbb{E}\left[\frac{|\langle h_k; \mathcal{I}_k \rangle_{\text{tr}}|^2 P(P\xi^2 + \sigma_{\text{L}}^2)}{P|\langle h_k; \mathcal{I}_k \rangle_{\text{tr}}|^2 + P\xi^2 + \sigma_{\text{L}}^2} \right], \tag{28}$$

where ξ^2 and $\langle h_k; \mathcal{I}_k \rangle_{\text{tr}}$ are defined as Equations (19) and (21), respectively.

The difference between the fixed-point equations (20) and (28) is in the last terms of their right-hand sides. The last term of the right-hand side of Equation (28) corresponds to the MSE of the LMMSE estimate of $b_{k,t}$ in the fading channel (25) with perfect CSI at the receiver.

3.3 Derivation of Claim 1

The derivation of Claim 1 consists of three parts: To analyze the equivalent channel (8) between the kth user and the corresponding decoder, to evaluate the posterior pdf (3), and to combine the two analyses in order to obtain the fixed-point equation (20). The replica analysis given by Guo and Verdú[6] is applicable to the first part, by regarding $h_k b_k$ as single data symbol. The analysis of the second part was given by Evans and Tse[8] on the basis of random matrix theory. We contribute to applying the replica analysis of Guo and Verdú to our case, and analyzing the last part.

We first provide an analytical expression of the equivalent channel (8) between the kth user and the corresponding decoder in the large-system limit.

Definition 2 We consider the single-user fading channel (10) for the kth user in the communication phase. The receiver consists of the original channel estimator, an MMSE detector, and the kth user's decoder. The MMSE detector estimates each data symbol $b_{k,t}$ from $z_{k,t}$ and the marginal posterior pdf $p(h_k|\mathcal{I})$, which is given via the marginalization of $p(\mathcal{H}|\mathcal{I})$ provided by the original channel estimator, and provides soft information about the data symbol $b_{k,t}$ to the kth user's decoder in the form of the posterior pdf of $b_{k,t}$ given $z_{k,t}$ and $\mathcal{I}$:

$$p(b_{k,t}|z_{k,t}, \mathcal{I}) = \frac{p(z_{k,t}|b_{k,t}, \mathcal{I}) p(b_{k,t})}{\int p(z_{k,t}|b_{k,t}, \mathcal{I}) p(b_{k,t}) db_{k,t}}, \tag{29}$$

with

$$p(z_{k,t}|b_{k,t},\mathcal{I}) = \int p(z_{k,t}|h_k, b_{k,t})p(h_k|\mathcal{I})dh_k. \tag{30}$$

Claim 2[6] *Let* $\underline{\tilde{b}_{k,t}} \in \mathbb{C}$ *denote a random variable following the posterior pdf* (29). *If Assumption* 1 *holds, the equivalent channel* (8) *between the kth user and the corresponding decoder converges in law to* $p(\underline{\tilde{b}_{k,t}}|b_{k,t},\mathcal{I})$, *defined in the same manner as Equation* (16), *in the large-system limit. Furthermore, the spectral efficiency* (6) *converges in law to the spectral efficiency*

$$\tilde{C}_{\mathrm{as}} = \lim_{K\to\infty} \frac{\beta}{KT_{\mathrm{c}}} \sum_{t=\tau+1}^{T_{\mathrm{c}}} \sum_{k=1}^{K} I(b_{k,t};\underline{\tilde{b}_{k,t}}|\mathcal{I}), \tag{31}$$

where the mutual information $I(b_{k,t};\underline{\tilde{b}_{k,t}}|\mathcal{I})$ *is defined as*

$$I(b_{k,t};\underline{\tilde{b}_{k,t}}|\mathcal{I}) = \int p(\underline{\tilde{b}_{k,t}}|b_{k,t},\mathcal{I})p(b_{k,t}) \log \frac{p(\underline{\tilde{b}_{k,t}}|b_{k,t},\mathcal{I})}{\int p(\underline{\tilde{b}_{k,t}}|b_{k,t},\mathcal{I})p(b_{k,t})db_{k,t}} db_{k,t} d\underline{\tilde{b}_{k,t}}. \tag{32}$$

The variances σ_t^2 *for* $t = \tau+1, \tau+2, \cdots, T_{\mathrm{c}}$, *which are included in* $p(\underline{\tilde{b}_{k,t}}|b_{k,t},\mathcal{I})$, *are given by* σ_{c}^2, *which is a solution of the fixed-point equation:*

$$\sigma_{\mathrm{c}}^2 = N_0 + \lim_{K\to\infty} \frac{\beta}{K} \sum_{k=1}^{K} \mathbb{E}\left[|h_k b_{k,t} - \langle h_k b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\mathrm{c}}|^2 |\mathcal{I}\right], \tag{33}$$

where the posterior mean $\langle h_k b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\mathrm{c}}$ *of* $h_k b_{k,t}$ *given* $z_{k,t}$ *and* $\mathcal{I}$ *is defined as*

$$\langle h_k b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\mathrm{c}} = \int h_k b_{k,t} p(h_k, b_k|z_{k,t},\mathcal{I}) dh_k db_{k,t}, \tag{34}$$

with

$$p(h_k, b_k|z_{k,t},\mathcal{I}) = \frac{p(z_{k,t}|h_k, b_{k,t})p(h_k|\mathcal{I})p(b_{k,t})}{\int p(z_{k,t}|h_k, b_{k,t})p(h_k|\mathcal{I})p(b_{k,t})dh_k db_{k,t}}. \tag{35}$$

If the fixed-point equation (33) *has multiple solutions, one should choose the solution to minimize the following quantity:*

$$\lim_{K\to\infty} \frac{\beta}{K} \sum_{k=1}^{K} I(b_{k,t};\underline{\tilde{b}_{k,t}}|\mathcal{I}) + \mathrm{D}(\mathcal{CN}(0,N_0)\|\mathcal{CN}(0,\sigma_{\mathrm{c}}^2)). \tag{36}$$

Claim 2 implies that the equivalent channel between the kth user and its decoder asymptotically looks like a single-user fading channel. The differences between it and the single-user fading channel defined in Definition 1 are two points: One is that the posterior pdf (11) is replaced by the original marginal posterior pdf $p(h_k|\mathcal{I})$ in the definition of the channel estimator, and the other is the definition of σ_{c}^2. We can reduce the fixed-point equation (33) to a more explicit formula.

Corollary 1 *The fixed-point equation* (33) *is explicitly given by*

$$\sigma_{\mathrm{c}}^2 = N_0 + \lim_{K\to\infty} \frac{\beta}{K} \sum_{k=1}^{K} \left\{ \frac{P\xi_k^2 \sigma_{\mathrm{c}}^2}{P\xi_k^2 + \sigma_{\mathrm{c}}^2} + \left(\frac{\sigma_{\mathrm{c}}^2}{P\xi_k^2 + \sigma_{\mathrm{c}}^2}\right)^2 |\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}|^2 \mathbb{E}\left[|b_{k,t} - \langle b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\mathrm{c}}|^2 \middle| \mathcal{I}\right] \right\}, \tag{37}$$

where $\langle b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\mathrm{c}}$ *denotes the posterior mean of* $b_{k,t}$ *conditioned on* $z_{k,t}$ *and* $\mathcal{I}$*, defined in the same manner as Equation* (22)*, and where* $\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}$ *and* ξ_k^2 *are the mean and variance of the original marginal posterior pdf* $p(h_k|\mathcal{I})$*, respectively.*

The proof of Corollary 1 is provided in Appendix. $\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}$ and ξ_k^2, which are random variables depending on $\mathcal{I}$, appear in Equation (37) because the original marginal posterior pdf $p(h_k|\mathcal{I})$ is Gaussian.

We next refer to the asymptotic analysis[8] of $\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}$ and ξ_k^2 in order to show that the fixed-point equation (37) coincides with the fixed-point equation (20).

Theorem 2[8] ξ_k^2 *converges in probability to the asymptotic MSE* ξ^2 *of the MMSE estimate of* h_k*, defined as Equation* (19)*, in the large-system limit for any* k*.* $\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}$ *converges in law to a circularly symmetric complex Gaussian random variable with variance* $1-\xi^2$ *in the same limit. Furthermore, the empirical pdf* $\rho_K(\lambda) = K^{-1}\sum_{k=1}^{K}\delta(\lambda - |\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}|^2)$ *of* $|\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}|^2$ *converges in probability to a deterministic pdf in the large-system limit.*

It is straightforward to confirm that the last term of the right-hand side of the fixed-point equation (37) depends on $\mathcal{I}$ only through $|\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}|^2$. The last statement in Theorem 2 implies that the fixed-point equation (37) converges in probability to Equation (20), due to the uniform statistical properties for all users and $\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}} \sim \langle h_k; \mathcal{I}_k\rangle_{\mathrm{tr}}$.

Our derivation is completed by showing that the spectral efficiency (31) converges in probability to the expectation of the spectral efficiency (14). We know that the mutual information (32) coincides with the mutual information $I(b_{k,t}; \langle b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\mathrm{c}}|\mathcal{I})$. It is straightforwardly shown that $I(b_{k,t}; \langle b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\mathrm{c}}|\mathcal{I})$ depends on $\mathcal{I}$ only via $|\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}}|^2$. Therefore, from Theorem 2, we find that the spectral efficiency (31) converges in probability to

$$\frac{\beta}{T_{\mathrm{c}}}\sum_{t=\tau+1}^{T_{\mathrm{c}}}\mathbb{E}[I(b_{k,t}; \langle b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\mathrm{c}}|\mathcal{I})], \tag{38}$$

which is equal to the expectation of the spectral efficiency (14) due to $\langle h_k; \mathcal{I}\rangle_{\mathrm{tr}} \sim \langle h_k; \mathcal{I}_k\rangle_{\mathrm{tr}}$.

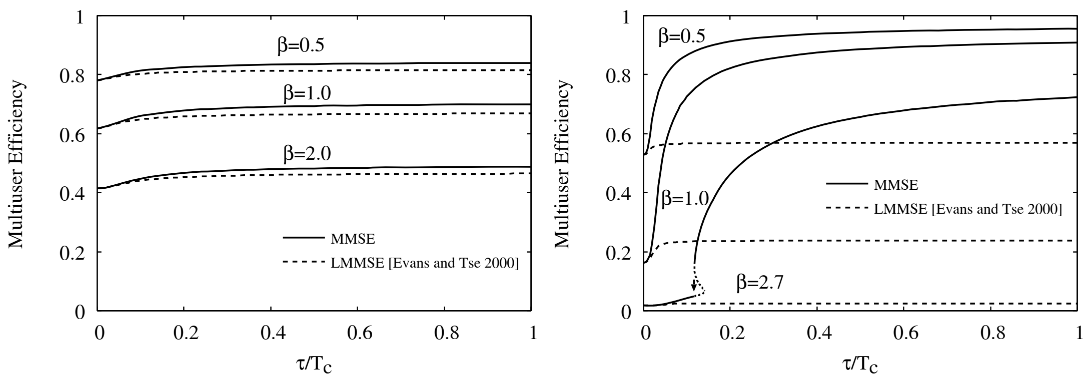

Figure 1 Multiuser efficiency versus τ/T_{c} for $T_{\mathrm{c}} = 40$ at $P/N_0 = 0$ dB (left) and $P/N_0 = 15$ dB (right). The solid and dashed lines represent the multiuser efficiencies of the MMSE and LMMSE multiuser detectors, respectively. The dotted curve in the right figure shows solutions for the MMSE multiuser detector which are not chosen by the criterion described in Claim 1

4 Numerical Examples

We numerically compare the spectral efficiency of the MMSE multiuser detector with that of the LMMSE multiuser detector on the basis of Claim 1 and Theorem 1. We first focus on the ratio N_0/σ_c^2 (N_0/σ_L^2) of the original noise variance to the effective noise variance, called multiuser efficiency, because the difference between the two spectral efficiencies results from that between their multiuser efficiencies. Figure 1 shows the multiuser efficiencies of the MMSE and LMMSE multiuser detectors. The gap between their multiuser efficiencies is small for any β at low signal-to-noise ratio (SNR) ($P/N_0 = 0$ dB). On the other hand, the gap at high SNR ($P/N_0 = 15$ dB) is negligible if τ/T_c is sufficiently small, but it becomes large as τ/T_c grows. Furthermore, the fixed-point equations (20) for the MMSE multiuser detector has multiple solutions when $\beta = 2.7$ and $P/N_0 = 15$ dB. The dotted curve shown in Figure 1 displays the solutions which is not chosen by the criterion described in Claim 1. This result indicates that the multiuser efficiency of the MMSE multiuser detector exhibits a waterfall behavior in this region: It falls down toward that of the LMMSE multiuser detector as τ/T_c becomes slightly smaller than a critical point, $\tau/T_c \approx 0.117$ for $\beta = 2.7$ and $P/N_0 = 15$ dB.

These results do not necessarily imply that the MMSE multiuser detector outperforms the LMMSE multiuser detector for large τ/T_c in terms of spectral efficiency because large τ/T_c means small payload. Figure 2 displays the spectral efficiencies of the MMSE and LMMSE multiuser detectors. The spectral efficiencies (17) and (26) increase as τ/T_c grows from zero because the accuracy of the channel estimation improves, and they are maximized at optimal values of τ. However, they decrease as τ/T_c grows beyond the optimal values due to the decrease of data symbols within a coherent interval. The gap between the spectral efficiencies of the MMSE and LMMSE multiuser detectors is also maximized in the neighborhood of the optimal values, and is negligibly small for any β at low SNR ($P/N_0 = 0$ dB). The spectral efficiency of the LMMSE multiuser detector is close to that of the MMSE multiuser detector at high SNR ($P/N_0 = 15$ dB) if β is small. These numerical results imply that the time-complexity of the MUD is significantly reduced in the large-system limit when P/N_0 is small or when P/N_0 is large and β is small, if small performance loss is tolerated.

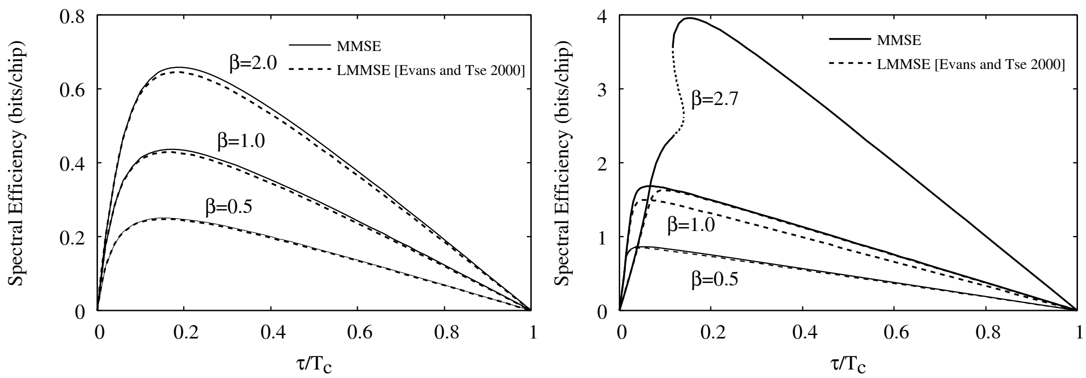

Figure 2 Spectral efficiency versus τ/T_c for $T_c = 40$ at $P/N_0 = 0$ dB (left) and $P/N_0 = 15$ dB (right). The solid and dashed lines represent the spectral efficiencies of the MMSE and LMMSE multiuser detectors, respectively. The dotted curve in the right figure shows solutions for the MMSE multiuser detector which are not chosen by the criterion described in Claim 1

5 Conclusions

We evaluated the asymptotic performance gap between the MMSE and LMMSE multiuser detectors in the randomly-spread CDMA channel, in which the fading process is not known to the base station. We found that the gap is negligibly small when the SNR is low or when the SNR is high and the system load is small. Our results imply that the time-complexity of MUD can be significantly reduced in the large-system limit in return for small performance loss even if the influence of channel estimation is considered, or to put it another way, that the performance of MUD scarcely improves by spending more computational cost than that of the LMMSE MUD if the random spreading, the large-system limit, and small system loads are assumed.

We note that, as discussed in Subsection 2.1, our MUD scheme does not utilize received outputs in different symbol periods for the detection of each data symbol. Furthermore, we remark that the assumptions of the QPSK modulation and i.i.d. Rayleigh fading are crucial in order to derive our results. It is a future work to relax these assumptions.

Appendix: Proof of Corollary 1

We first calculate the conditional joint pdf $p(z_{k,t}, h_k|b_{k,t}, \mathcal{I})$ in order to evaluate the posterior mean (34) of $h_k b_{k,t}$ given $z_{k,t}$ and $\mathcal{I}$. h_k conditioned on $\mathcal{I}$ is Gaussian because we have assumed $h_k \sim \mathcal{CN}(0,1)$,

$$p(h_k|\mathcal{I}) = \frac{1}{\pi\xi_k^2}\exp\left(-\frac{|h_k - \langle h_k; \mathcal{I}\rangle_{\rm tr}|^2}{\xi_k^2}\right). \tag{39}$$

On the other hand, Equation (10) yields

$$p(z_{k,t}|h_k, b_{k,t}) = \frac{1}{\pi\sigma_{\rm c}^2}\exp\left(-\frac{|z_{k,t} - h_k b_{k,t}|^2}{\sigma_{\rm c}^2}\right) \quad \text{for } t > \tau. \tag{40}$$

Calculating $p(z_{k,t}, h_k|b_{k,t}, \mathcal{I}) = p(z_{k,t}|h_k, b_{k,t})p(h_k|\mathcal{I})$, after some algebra, we obtain

$$p(z_{k,t}, h_k|b_{k,t}, \mathcal{I}) = p(h_k|z_{k,t}, b_{k,t}, \mathcal{I})p(z_{k,t}|b_{k,t}, \mathcal{I}), \tag{41}$$

with

$$p(h_k|z_{k,t}, b_{k,t}, \mathcal{I}) = \frac{P\xi_k^2 + \sigma_{\rm c}^2}{\pi\xi_k^2\sigma_{\rm c}^2}{\rm e}^{-\frac{P\xi_k^2+\sigma_{\rm c}^2}{\xi_k^2\sigma_{\rm c}^2}\left|h_k - \frac{\xi_k^2}{P\xi_k^2+\sigma_{\rm c}^2}b_{k,t}^* z_{k,t} - \frac{\sigma_{\rm c}^2}{P\xi_k^2+\sigma_{\rm c}^2}\langle h_k;\mathcal{I}\rangle_{\rm tr}\right|^2}, \tag{42}$$

where $b_{k,t}^*$ denotes the complex conjugate of $b_{k,t}$, and where we have used the fact that $|b_{k,t}|^2$ equals P with probability one. By definition, the posterior mean (34) is given by

$$\begin{aligned}\langle h_k b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\rm c} &= \frac{\int h_k b_{k,t} p(z_{k,t}, h_k|b_{k,t}, \mathcal{I})p(b_{k,t})dh_k db_{k,t}}{\int p(z_{k,t}, h_k|b_{k,t}, \mathcal{I})p(b_{k,t})dh_k db_{k,t}} \\ &= \frac{P\xi_k^2}{P\xi_k^2 + \sigma_{\rm c}^2}z_{k,t} + \frac{\sigma_{\rm c}^2}{P\xi_k^2 + \sigma_{\rm c}^2}\langle h_k; \mathcal{I}\rangle_{\rm tr}\langle b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\rm c}. \end{aligned} \tag{43}$$

We next evaluate the fixed-point equation (33). Substituting Equation (43) into the second term of the right-hand side of Equation (33), we obtain

$$\begin{aligned}&\mathbb{E}\left[|h_k b_{k,t} - \langle h_k b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\rm c}|^2 \middle| \mathcal{I}\right] \\ &= \mathbb{E}\left[a_{k,t} + \frac{\sigma_{\rm c}^2}{P\xi_k^2 + \sigma_{\rm c}^2}\langle h_k; \mathcal{I}\rangle_{\rm tr}(b_{k,t} - \langle b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\rm c})\middle| \mathcal{I}\right], \end{aligned} \tag{44}$$

where $a_{k,t}$ is defined as

$$a_{k,t} = \frac{\sigma_{\rm c}^2}{P\xi_k^2 + \sigma_{\rm c}^2}(h_k - \langle h_k; \mathcal{I}\rangle_{\rm tr})b_{k,t} - \frac{P\xi_k^2}{P\xi_k^2 + \sigma_{\rm c}^2}v_{k,t}. \tag{45}$$

The second term in the expectation of the right-hand side of Equation (44) is a function of $z_{k,t}$ conditioned on $b_{k,t}$ and $\mathcal{I}$. The fact that $a_{k,t}$ and $z_{k,t}$ are jointly Gaussian conditioned on $b_{k,t}$ and $\mathcal{I}$ is useful in order to show that the two terms in the expectation of the right-hand side of Equation (44) are mutually independent conditioned on $b_{k,t}$ and $\mathcal{I}$. The means of $a_{k,t}$ and $z_{k,t}$ conditioned on $b_{k,t}$ and $\mathcal{I}$ are zero and $\langle h_k; \mathcal{I} \rangle_{\rm tr} b_{k,t}$, respectively, and the covariance matrix of the column vector $(a_{k,t}, z_{k,t})^{\rm T}$ conditioned on $b_{k,t}$ and $\mathcal{I}$ is evaluated as the diagonal matrix $\mathrm{diag}\{P\xi_k^2\sigma_{\rm c}^2/(P\xi_k^2+\sigma_{\rm c}^2),\ P\xi_k^2+\sigma_{\rm c}^2\}$. Therefore, $a_{k,t}$ and $z_{k,t}$ are mutually independent conditioned on $b_{k,t}$ and $\mathcal{I}$. This fact indicates that Equation (44) yields

$$\begin{aligned}&\mathbb{E}\left[|h_k b_{k,t} - \langle h_k b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\rm c}|^2 | \mathcal{I}\right] \\ &= \frac{P\xi_k^2\sigma_{\rm c}^2}{P\xi_k^2+\sigma_{\rm c}^2} + \left(\frac{\sigma_{\rm c}^2}{P\xi_k^2+\sigma_{\rm c}^2}\right)^2 |\langle h_k; \mathcal{I}\rangle_{\rm tr}|^2 \mathbb{E}\left[|b_{k,t} - \langle b_{k,t}; z_{k,t}, \mathcal{I}\rangle_{\rm c}|^2 \middle| \mathcal{I}\right]. \end{aligned} \tag{46}$$

Substituting Equation (46) into the fixed-point equation (33), we obtain Equation (37).

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