A Random Matrix Model for Communication via Antenna Arrays

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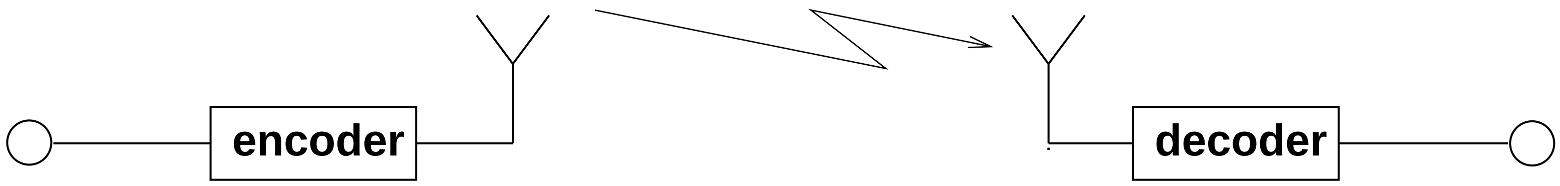
1. Motivation
2. Channel Model
3. Free Random Variables
4. Performance Analysis
5. Inter–Symbol Interference
6. Design of Low–Complexity Receivers

Information Theory of Communications



Electro–magnetic waves via empty space:

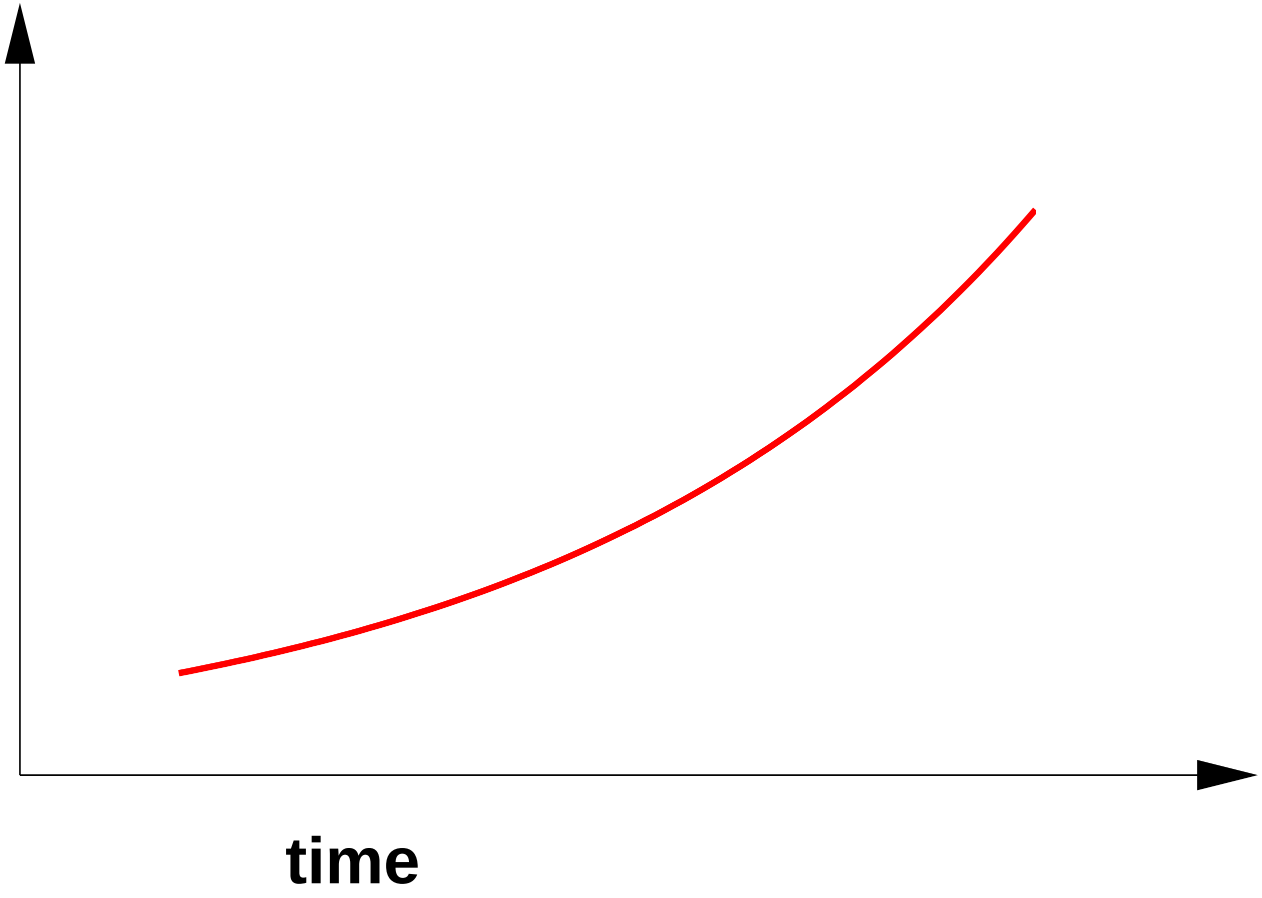
$$C = B \cdot \log_2\left(1 + \frac{E_b}{N_0} \cdot \frac{C}{B}\right)$$

bit rate ≈ bandwidth · logarithm of energy

bit/s Hz bit

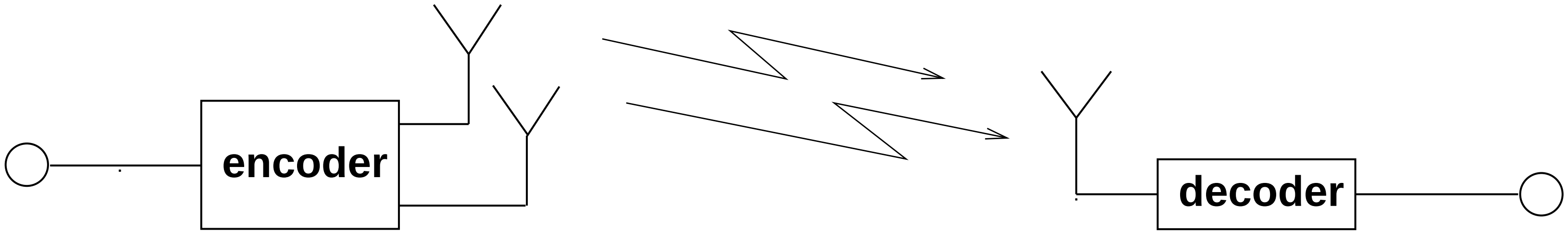
C. Shannon ('48)

Shannon vs. Moore



- bandwidth is expensive
- energy does not really help

Multiple–Element Antennas



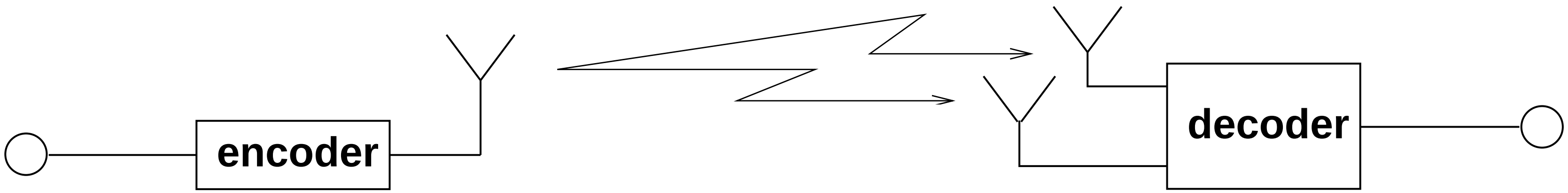
Electro–magnetic waves via empty space:

$$C \quad = \quad B \quad \cdot \log_2\left(1 + \frac{2E_{\mathrm{b}}}{N_0} \cdot \frac{C}{B}\right)$$

bit rate ≈ bandwidth · logarithm of energy

bit/s Hz bit

Multiple–Element Antennas (cont'd)



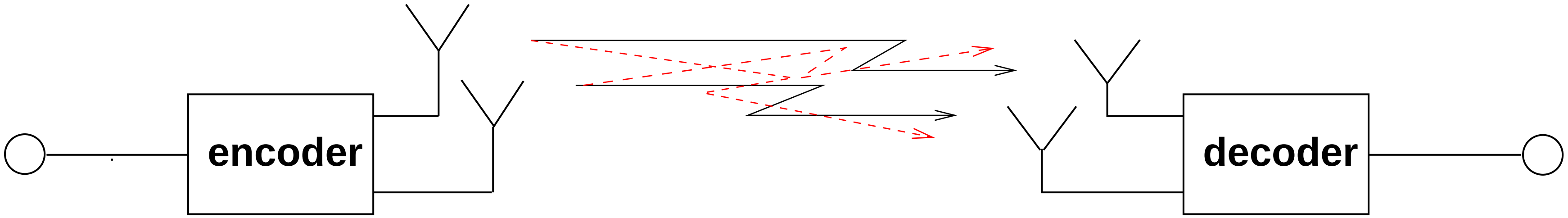
Electro–magnetic waves via empty space:

$$C = B \cdot \log_2 \left(1 + \frac{2E_\mathrm{b}}{N_0} \cdot \frac{C}{B} \right)$$

bit rate ≈ bandwidth · logarithm of energy

bit/s Hz bit

Multiple–Element Antennas at *Both Ends*



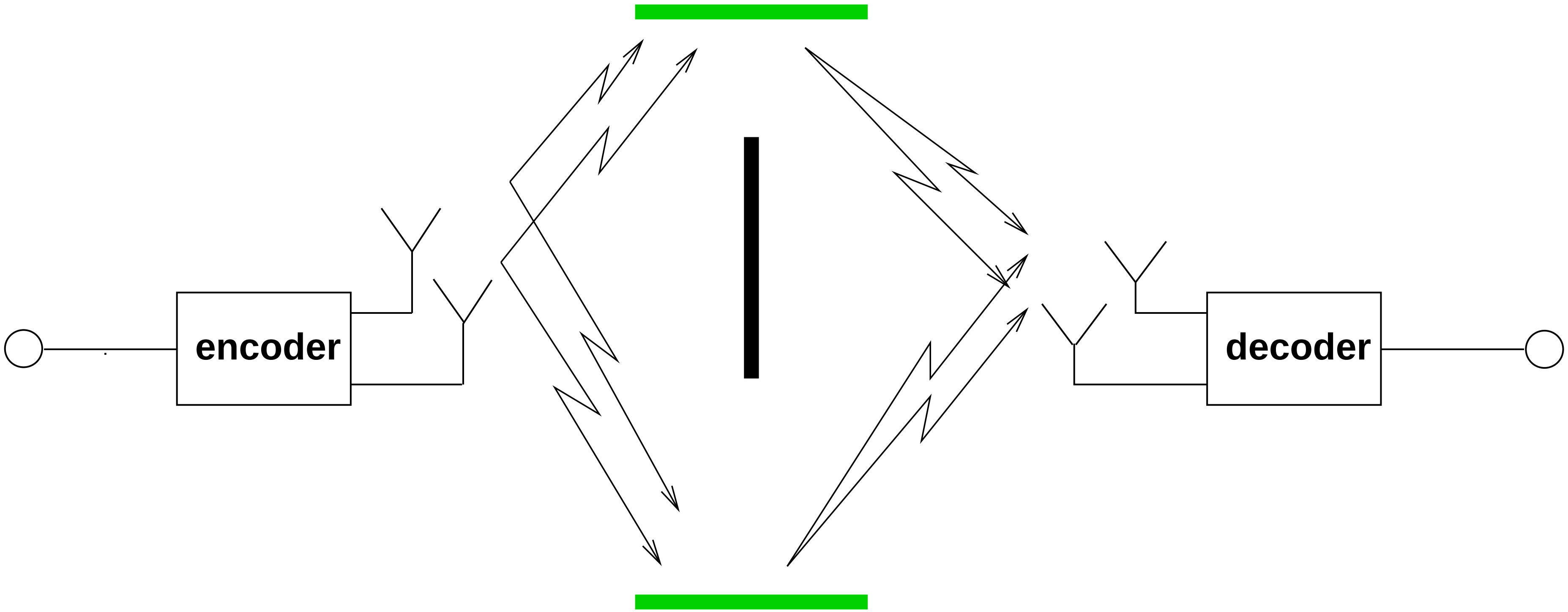
Electro–magnetic waves via empty space:

$$C = B \cdot \log_2 \det \left(\mathbf{I} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{H} \frac{E_{\mathrm{b}}}{N_0} \cdot \frac{C}{B} \right)$$

$$= 2B \cdot \log_2 \left(1 + \frac{E_{\mathrm{b}}}{N_0} \cdot \frac{C}{2B} \right) \quad \text{for} \quad \boldsymbol{H} = \mathbf{I}$$

Foschini '96, Paulraj & Kailath '94, (Verdú '86)

The Key Ingredient: Rich Scattering



How to characterize such a channel?

Algebraic Characterization

Linear multiple–input multiple–output (MIMO) system described by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_R \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \ldots & h_{1T} \\ h_{21} & h_{22} & h_{23} & \ldots & h_{2T} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{R1} & h_{R2} & h_{R3} & \ldots & h_{RT} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_T \end{bmatrix}$$

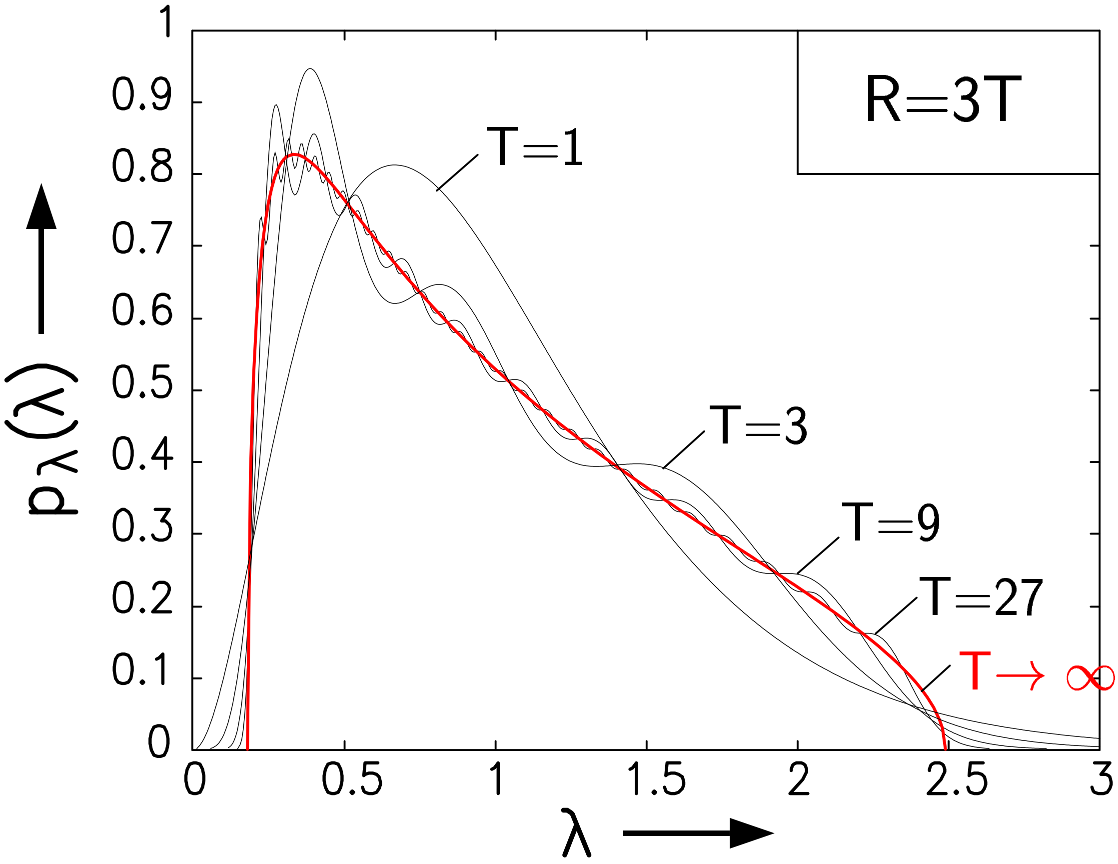
$R \times 1$	$R \times T$	$T \times 1$
receiver	channel	transmitter

Standard Model

Assume entries of $\boldsymbol{H}$

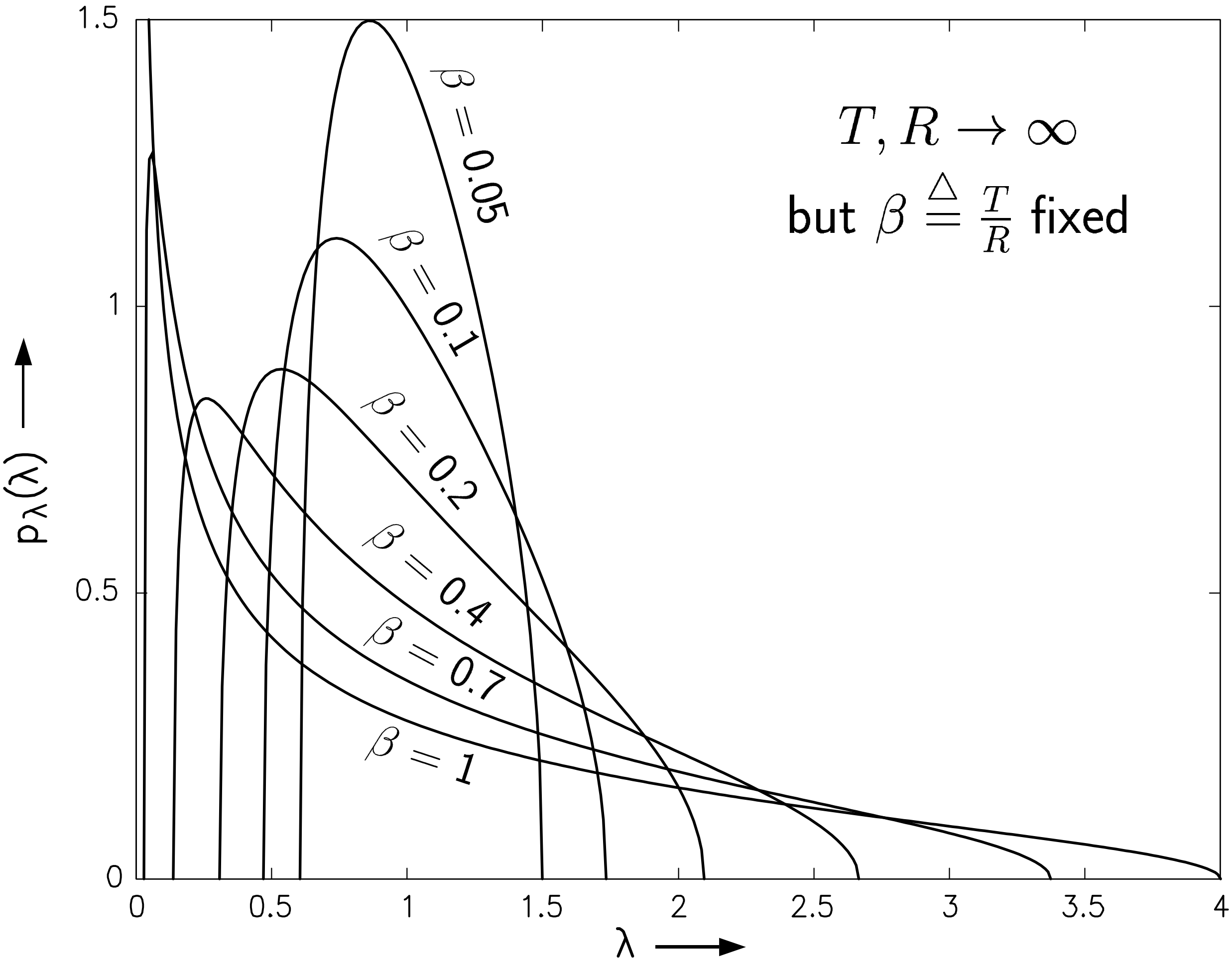
- zero–mean
- independent
- complex Gaussian distributed.

$\lambda_1, \ldots, \lambda_T$: eigenvalues of $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$



Closed form for eigenvalue density.

Standard Model cont'd



Trouble:
Dependencies among entries of $\boldsymbol{H}$ due to

- limited number of scatterers
- correlation between closely spaced antennas

$\Downarrow$

Need for a new model

Proposed Model

to account for limited scattering

$$\boldsymbol{H} = \sqrt{P}\,\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Theta}$$

$$\boldsymbol{H} = \sqrt{\frac{P}{RS}} \begin{bmatrix} e^{j\varphi_{11}} & e^{j\varphi_{12}} & \dots & e^{j\varphi_{1S}} \\ e^{j\varphi_{21}} & e^{j\varphi_{22}} & \dots & e^{j\varphi_{2S}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j\varphi_{R1}} & e^{j\varphi_{R2}} & \dots & e^{j\varphi_{RS}} \end{bmatrix} \begin{bmatrix} e^{j\vartheta_{11}} & e^{j\vartheta_{12}} & \dots & e^{j\vartheta_{1T}} \\ e^{j\vartheta_{21}} & e^{j\vartheta_{22}} & \dots & e^{j\vartheta_{2T}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j\vartheta_{S1}} & e^{j\vartheta_{S2}} & \dots & e^{j\vartheta_{ST}} \end{bmatrix}$$

$R \times T$	$R \times S$	$S \times T$
channel	receive array	transmit array

S : number of scattering objects

$\sqrt{P}$: path gain

$\varphi_{\cdot}, \vartheta_{\cdot}$: independent uniformly distributed phases

Asymptotic Eigenvalue Distribution

Theorem: *Let* $\boldsymbol{C} \triangleq \boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ *and* $\boldsymbol{\mathcal{L}}$ *be the set containing the eigenvalues of* $\boldsymbol{C}$*. Then, for i.i.d. zero–mean (Gaussian) random entries of* $\boldsymbol{\Phi}$ *and* $\boldsymbol{\Theta}$ *with finite variance the eigenvalue distribution*

$$\mathrm{P}_{\boldsymbol{C}}(x) \triangleq \frac{1}{T}\Big|\{\lambda \in \boldsymbol{\mathcal{L}} : \lambda < x\}\Big|$$

converges to a fixed non–random *distribution if* $R, T, S \to \infty$*, but*

$$\beta \triangleq \frac{T}{R} \qquad \text{and} \qquad \rho \triangleq \frac{S}{R}$$

system load — *richness*

remain constant.

Convergence holds for the eigenvalues of a single random matrix (self–averaging p...)

Non–Commutative Probability Spaces

Classical Probability

Independence means:
$\mathrm{E}\{A^m B^n\} = \mathrm{E}\{A^m\}\,\mathrm{E}\{B^n\}$
for all m, n, e.g.
$\mathrm{E}\{ABAB\} = \mathrm{E}\{A^2\}\,\mathrm{E}\{B^2\}$
impossible for general matrices!

Non–Commutative Probability

Freeness means:
$\phi\left\{A^{i_1} B^{i_2} A^{i_3} B^{i_4} \cdots\right\} = 0$
whenever
$\phi\left\{A^{i_n}\right\} = 0$ or $\phi\left\{B^{i_n}\right\} = 0$

E.g.

$$\begin{aligned}
\phi(AB) &= \phi(A)\,\phi(B) \\
\phi(AABB) &= \phi\left(A^2\right)\phi\left(B^2\right) \\
\phi(ABAB) &= \phi\left(A^2\right)\phi^2(B) + \phi^2(A)\phi\left(B^2\right) - \phi^2(A)\phi^2(B)
\end{aligned}$$

Asymptotic Freeness of Random Matrices

Theorem [Voiculescu]: *Let*

$$\phi(\cdot) = \lim_{T\to\infty} \mathrm{E}\,\frac{1}{T}\,\mathrm{tr}\,(\cdot),$$

then random matrices with i.i.d. zero–mean Gaussian entries with finite variance are free as their dimensions grow towards infinity.

$$\Longrightarrow \qquad \phi(\boldsymbol{C}^n) = \lim_{T\to\infty} \mathrm{E}\,\frac{1}{T}\,\mathrm{tr}\,(\boldsymbol{C}^n) = \mathrm{E}\,\{\lambda^n\}$$

Recent results [Thorbjørnsen] show that the expectation operator E is not necessary, but convergence is almost surely.

The Stieltjes Transform

More convenient representation of eigenvalue distribution by Stieltjes transform:

$$G_{\boldsymbol{C}}(s) \triangleq \int \frac{1}{x+s} \, \mathrm{dP}_{\boldsymbol{C}}(x)$$

with s denoting the complex transform domain.

For the standard model (SM), it is known that

$$s\beta G_{\mathrm{SM}}^2(s) + \big(s + 1 - \beta\big) G_{\mathrm{SM}}(s) - 1 = 0.$$

The R–Transform (Cumulant Gen. Function)

$$R(w) \triangleq \frac{1}{w} - G^{-1}(w)$$

with

$$G^{-1}\big(G(w)\big) = w$$

and w denoting the complex transform domain.

Additive free convolution

$$\boldsymbol{Z} = \boldsymbol{X} + \boldsymbol{Y}$$

$$\Downarrow$$

$$R_{\boldsymbol{Z}}(w) = R_{\boldsymbol{X}}(w) + R_{\boldsymbol{Y}}(w)$$

For the standard model, we get

$$R_{\mathrm{SM}}(w) = \frac{1}{\beta w + 1}.$$

The S–Transform

$$S(z) \triangleq \frac{1+z}{z}\Upsilon^{-1}(z)$$

with

$$\Upsilon(s) = \frac{G(-s^{-1})}{-s} - 1$$

and z denoting the complex transform domain.

Multiplicative free convolution

$$\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{Y}$$

$$\Downarrow$$

$$S_{\boldsymbol{Z}}(z) = S_{\boldsymbol{X}}(z)S_{\boldsymbol{Y}}(z)$$

For the standard model, we get

$$S_{\mathrm{SM}}(z) = \frac{1}{\beta z + 1}.$$

Analysis of Proposed Model

$$\boldsymbol{C} = P\boldsymbol{\Theta}^{\mathrm{H}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Theta} \qquad \rightsquigarrow \qquad \tilde{\boldsymbol{C}} = P\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Theta}\boldsymbol{\Theta}^{\mathrm{H}}$$

Non–zero eigenvalues of $\boldsymbol{C}$ and $\tilde{\boldsymbol{C}}$ are identical

$$TG_{\boldsymbol{C}}(s) - \frac{T}{s} = SG_{\tilde{\boldsymbol{C}}}(s) - \frac{S}{s}.$$

With system load β and richness ρ, it follows via S-transform

$$s^2 P\beta^2 G_{\boldsymbol{C}}^3(s) + sP\beta\big(\rho + 1 - 2\beta\big)G_{\boldsymbol{C}}^2(s) + \big(s\rho + P(\beta - \rho)(\beta - 1)\big)G_{\boldsymbol{C}}(s) - \rho = 0.$$

Performance Analysis

Antenna array system

$$\boldsymbol{y}_k = \boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{n}_k$$

with uncorrelated AWGN $\boldsymbol{n}$ of variance σ^2.

Similar to BLAST (Bell–Labs Layered Space–Time)

Assumptions:

- No antenna coupling
- (No intersymbol interference)
- Scattering follows proposed model

Performance measures:

- SINR with Wiener filter
- SINR with nulling
- Channel capacity

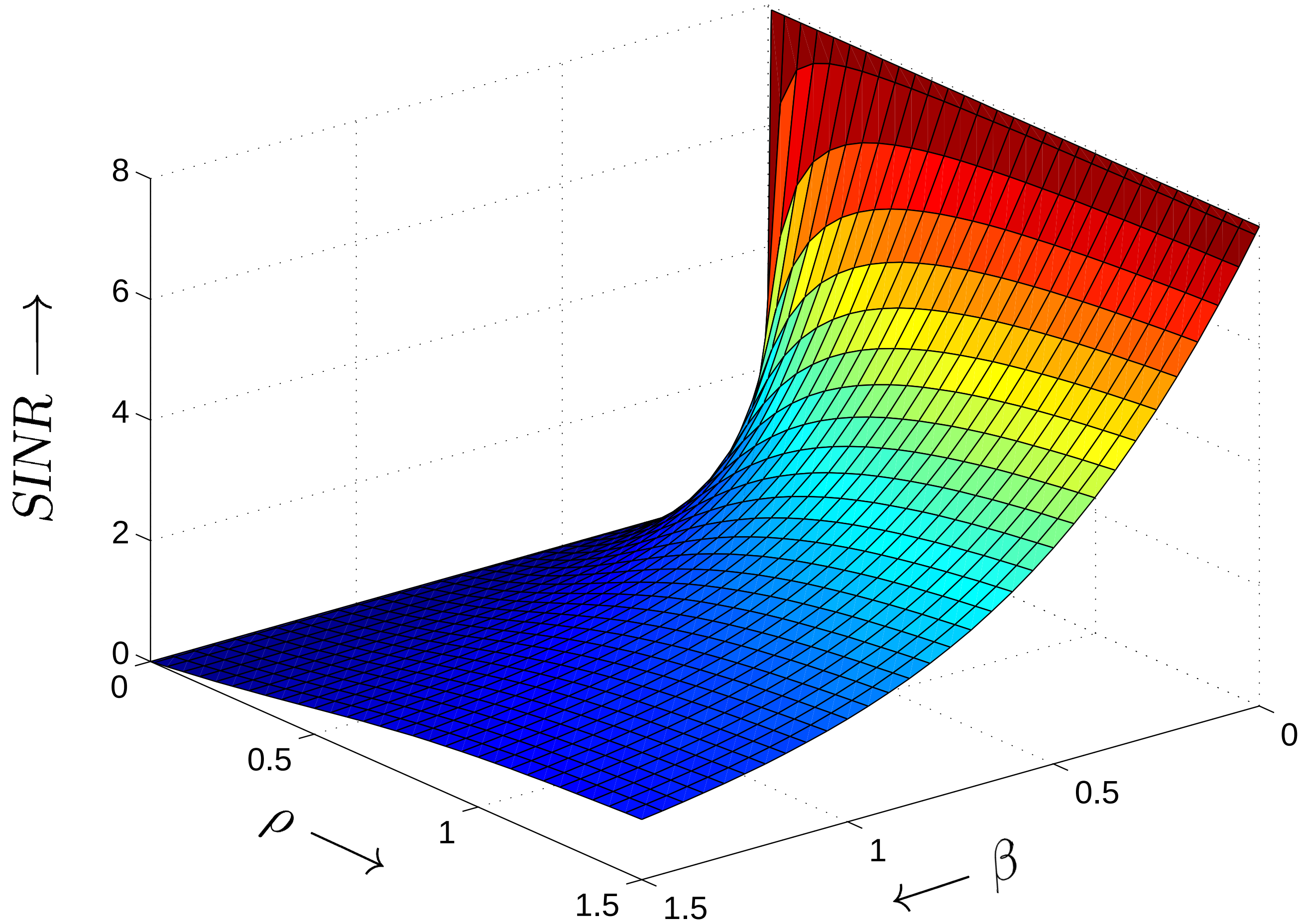
Optimum Linear Detection

(Wiener Filtering, Linear MMSE Equalization, Capon's beam former)

$$\hat{\boldsymbol{x}}_k = \left(\boldsymbol{C} + \sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \left(\boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{n}_k\right)$$

SINR can be given directly in terms of Stieltjes transforms:

$$SINR = \frac{1}{\sigma^2 G_{\boldsymbol{C}}(\sigma^2)} - 1$$



Richness ρ should be larger than the system load β.

Zero–Forcing Detection

(Nulling, Channel inversion, Decorrelation)

$$\hat{\boldsymbol{x}}_k = \boldsymbol{C}^{-1}\boldsymbol{H}^{\mathrm{H}}\left(\boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{n}_k\right)$$

$$SINR^{\mathrm{zf}} = \frac{1}{\sigma^2 G_{\boldsymbol{C}}(0)} = \frac{P}{\sigma^2}\left(1-\zeta\right)\left(1-\beta\right) \qquad \forall\, \beta, \zeta \leq 1$$

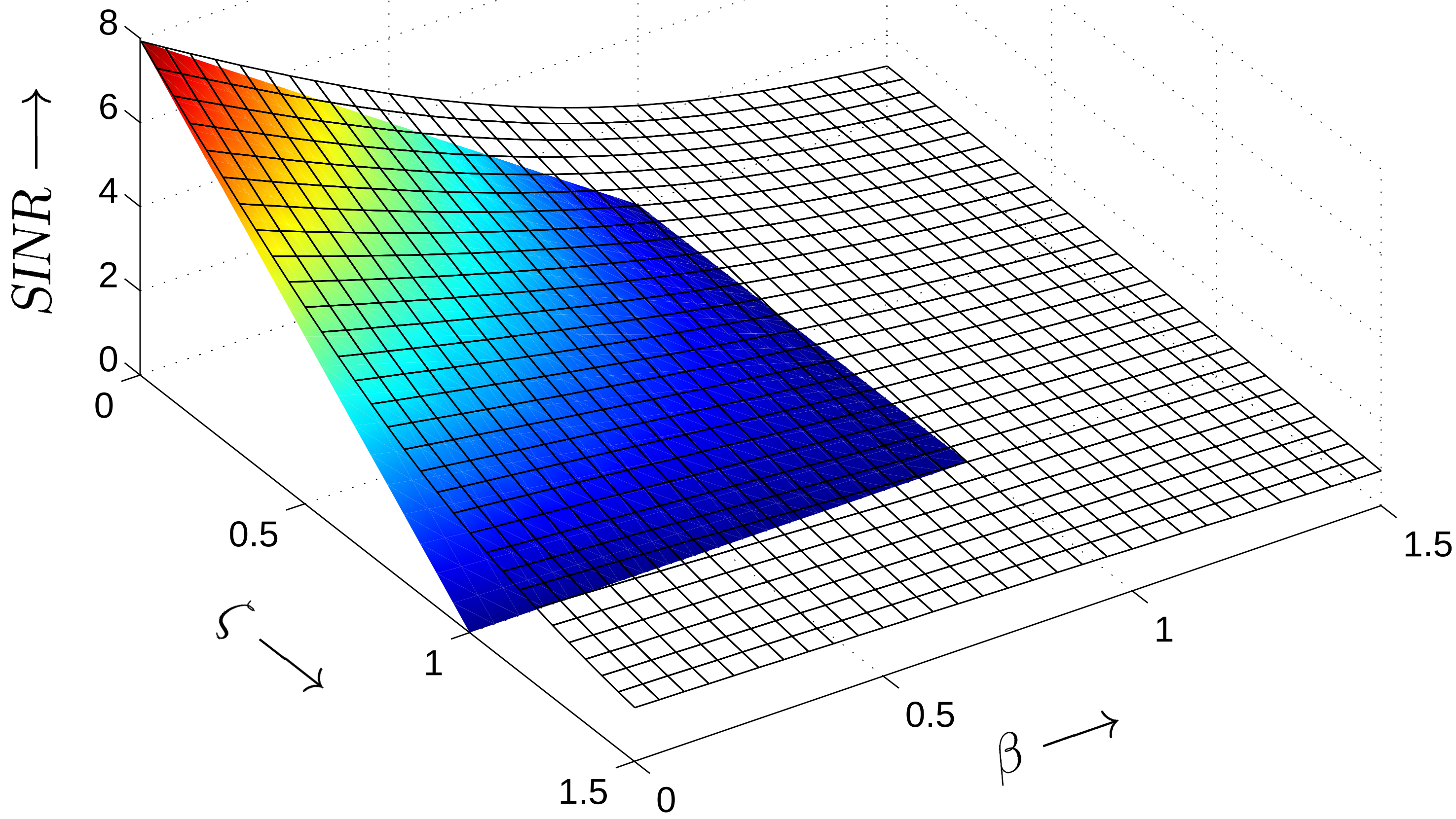
with the channel load

$$\zeta \triangleq \frac{\beta}{\rho} = \frac{T}{S}.$$

Symmetry between system load β and *channel load* ζ in Stieltjes transform:

$$s^2 P\beta\zeta G^3_{\boldsymbol{C}}(s) + sP\big(\beta+\zeta-2\beta\zeta\big)G^2_{\boldsymbol{C}}(s) + \big(s + P(\beta-1)(\zeta-1)\big)G_{\boldsymbol{C}}(s) - 1 = 0.$$

Optimum Linear Detection vs. Zero–Forcing



"Capacity" Transform

Define channel capacity in nats per unit number of receive antennas as a function of the formal noise variance s

$$C_{\boldsymbol{C}}(s) = \beta \int \ln\left(1 + \frac{x}{s}\right) \mathrm{dP}_{\boldsymbol{C}}(x)$$

Then,

$$\frac{\partial}{\partial s} C_{\boldsymbol{C}}(s) = \beta \int \frac{1}{s+x} \mathrm{dP}_{\boldsymbol{C}}(x) - \frac{\beta}{s}$$

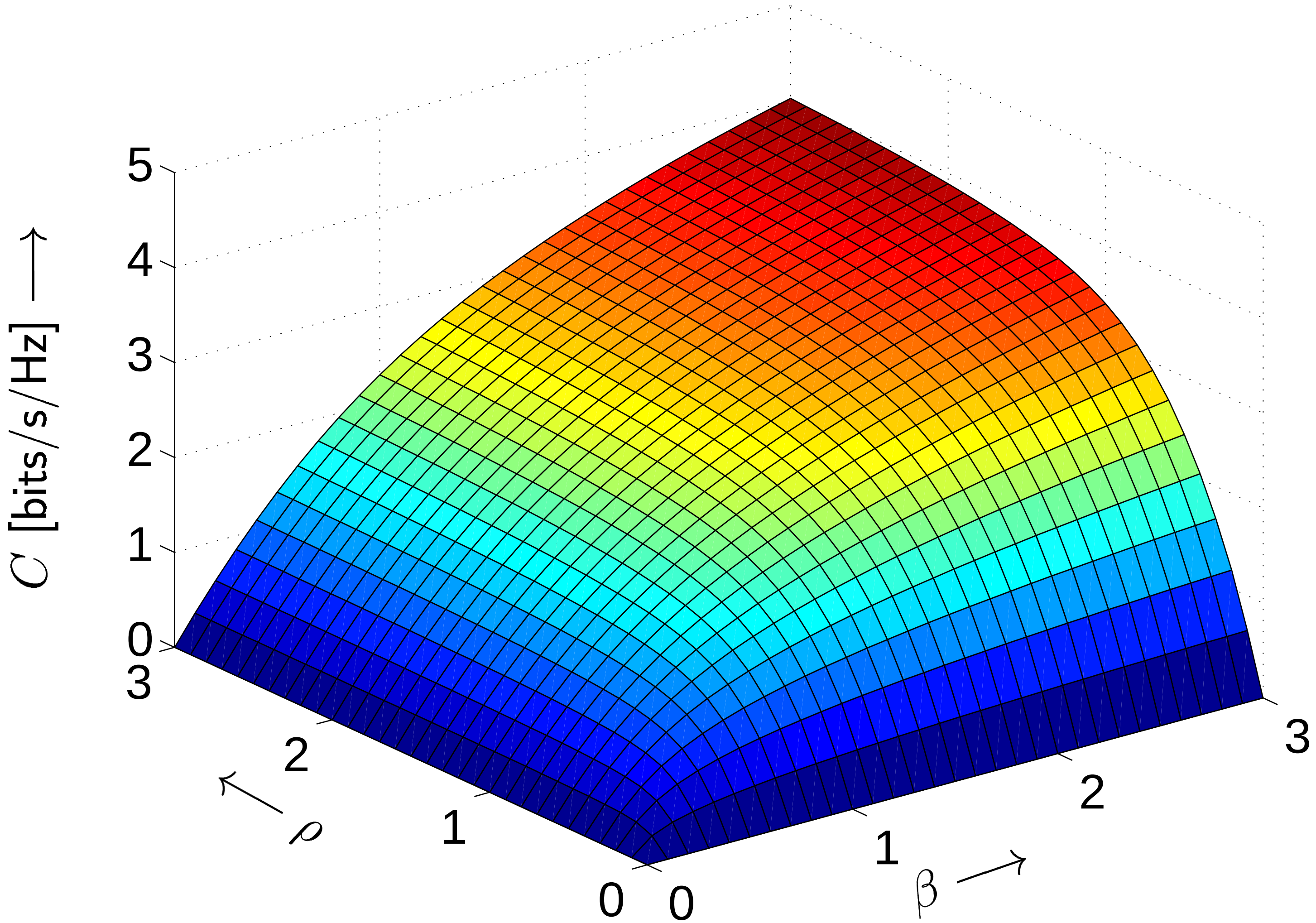
yields

$$G_{\boldsymbol{C}}(s) = \frac{1}{s} + \frac{1}{\beta} \frac{\partial}{\partial s} C_{\boldsymbol{C}}(s)$$

and

$$\left(s\frac{\partial C}{\partial s}\right)^3 + (\beta + \rho + 1)\left(s\frac{\partial C}{\partial s}\right)^2 + \left(\frac{s\rho}{P} + \rho + \beta\rho + \beta\right)\left(s\frac{\partial C}{\partial s}\right) + \beta\rho = 0.$$

Channel Capacity for $10\log_{10}\left(\frac{P}{\sigma^2}\right) = 9$ dB



Intersymbol Interference

Vector–valued time dispersive channel model

$$\boldsymbol{y}_k = \sum_{\ell=0}^{L-1} \boldsymbol{H}_\ell \, \boldsymbol{x}_{k-\ell}$$

Matrix notation

$$\begin{bmatrix} \vdots \\ \boldsymbol{y}_0 \\ \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ \cdots & \mathbf{0} & \boldsymbol{H}_{L-1} & \cdots & \boldsymbol{H}_1 & \boldsymbol{H}_0 & \mathbf{0} & \cdots & & \\ & \cdots & \mathbf{0} & \boldsymbol{H}_{L-1} & \cdots & \boldsymbol{H}_1 & \boldsymbol{H}_0 & \mathbf{0} & \cdots & \\ & & \cdots & \mathbf{0} & \boldsymbol{H}_{L-1} & \cdots & \boldsymbol{H}_1 & \boldsymbol{H}_0 & \mathbf{0} & \cdots \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}}_{\triangleq \boldsymbol{\mathcal{H}}} \begin{bmatrix} \vdots \\ \boldsymbol{x}_0 \\ \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \end{bmatrix}$$

Spatial and Temporal Multipath

Let S_ℓ be the number of scatterers corresponding to delay ℓ.
Similar

$$\begin{aligned} \rho &\rightsquigarrow \rho_\ell \\ \zeta &\rightsquigarrow \zeta_\ell \\ P &\rightsquigarrow P_\ell \end{aligned}$$

Theorem: *Let*

$$\boldsymbol{\Sigma} \triangleq \sum_{\ell=0}^{L-1} \boldsymbol{H}_\ell$$

and $R, T, S_\ell \to \infty$, *but* β, ρ_ℓ *remain constant, then the asymptotic eigenvalue distributions of*

$$\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}} \quad \text{and} \quad \boldsymbol{\Sigma}^{\mathrm{H}}\boldsymbol{\Sigma}$$

become identical.

There exists an equivalent non–dispersive channel.

Proof

$$\mathcal{H} = \mathcal{T}_R \mathcal{D} \mathcal{T}_T^{\mathrm{H}} \qquad (\mathcal{H} \text{ is block–circulant})$$

$\mathcal{T}_i$: block Fourier matrix on $i \times i$ identity matrices

$\mathcal{D}$: block diagonal matrix on $R \times T$ matrices

$$\mathcal{H} \sim \mathcal{D} \qquad \text{(eigenvalues are equivalent)}$$

blocks of $\mathcal{D}$ are given as

$$\sum_{\ell=0}^{L-1} \boldsymbol{H}_\ell \exp(\mathrm{j}\alpha_{i,\ell}) \sim \sum_{\ell=0}^{L-1} \boldsymbol{H}_\ell = \boldsymbol{\Sigma}$$

Total Richness

$$\sum_{\ell=0}^{L-1} \boldsymbol{H}_\ell = \left[\boldsymbol{\Phi}_0^{\mathrm{H}} \middle| \boldsymbol{\Phi}_1^{\mathrm{H}} \middle| \dots \middle| \boldsymbol{\Phi}_{L-1}^{\mathrm{H}}\right] \begin{bmatrix} P_0 \mathbf{I}_{S_0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & P_1 \mathbf{I}_{S_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & P_{L-1} \mathbf{I}_{S_{L-1}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Theta}_0 \\ \hline \boldsymbol{\Theta}_1 \\ \hline \vdots \\ \hline \boldsymbol{\Theta}_{L-1} \end{bmatrix}$$

For constant power delay profile $P_\ell = P$, multiple delays are equivalent to unit delay with total richness

$$\rho \triangleq \sum_{\ell=0}^{L-1} \rho_\ell$$

Equivalence of spatial and temporal multipath propagation.

Arbitrary Path Gains

For large number of antennas, the communication link is sufficiently characterized by

- the system load β,
- the total richness ρ,
- the distribution of the multiset of path gains in which each path gain P_ℓ is contained S_ℓ times

Comparison to Measurements

Indoor measurement at 2 GHz with 120 MHz bandwidth (no line of sight):

$R = T = 8$	$\overline{\lambda}$	$\overline{\lambda^2}$	$\overline{\lambda^3}$	$\overline{\lambda^4}$	$\overline{\lambda^5}$	$\overline{\lambda^6}$	$\overline{\lambda^7}$	$\overline{\lambda^8}$
no tin foil	1.00	3.32	14.7	74.2	406	2.35k	14.0k	86.5k
calc. $S = 6$	1.00	3.33	14.8	75.0	413	2.39k	14.4k	88.8k
with tin foil	1.00	2.34	6.96	23.0	81.4	301	1.16k	4.56k
calc. $S = 30$	1.00	2.27	6.67	22.3	80.8	307	1.21k	4.92k
Telatar mod.	1.00	2.00	5.02	14.2	43.1	139	466	1.62k
$S \to \infty$	1.00	2.00	5.00	14.0	42.0	132	429	1.43k

measured moments: $$\int_{1.94\,\mathrm{GHz}}^{2.06\,\mathrm{GHz}} \mathrm{trace}\left(\mathbf{H}^{\mathrm{H}}(f)\mathbf{H}(f)\right)^k \mathrm{d}f$$

SISO versus MIMO

For SISO channels, eigenvalues depend on the channel's impulse response, for MIMO channels they depend on the power delay profile and the number of scattering objects.

Receiver can obtain knowledge about the channel's eigenvalues more easily.

General Matrix Inversion

Let $\boldsymbol{C}$ be a non–singular square matrix.

Let λ_i denote the eigenvalues of $\boldsymbol{C}$.

Then,

$$\prod_{i=1}^{T} \left(\boldsymbol{C} - \lambda_i \mathbf{I} \right) = 0 \qquad \Longrightarrow \qquad \sum_{i=0}^{T} \alpha_i \boldsymbol{C}^{i} = 0$$

Cayley–Hamilton Theorem — with appropriate α_i s.

Solution to matrix inversion problem given the eigenvalues:

$$\boldsymbol{C}^{-1} = -\sum_{i=1}^{T} \frac{\alpha_i}{\alpha_0} \boldsymbol{C}^{i-1}$$

The Polynomial Expansion Receiver

Linear MMSE filter: $\boldsymbol{\mathcal{F}} = \left(\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}} + N_0\mathbf{I}\right)^{-1}\boldsymbol{\mathcal{H}}^{\mathrm{H}}$

Approximation by power series [Moshavi et al. '96]:

Cayley–Hamilton theorem yields:

$$\left(\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}} + N_0\mathbf{I}\right)^{-1} = \sum_{i=0}^{T-1} \tilde{w}_i \left(\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right)^i$$

$$\approx \sum_{i=0}^{D} w_i \left(\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right)^i \qquad \text{for } D < T-1.$$

How to choose the weights in real–time?

Weight Design

is given by Yule–Walker equations (Moshavi et al. '96):

$$\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{D+1} \end{bmatrix} = \begin{bmatrix} m_2+\sigma^2 m_1 & m_3+\sigma^2 m_2 & \dots & m_{D+2}+\sigma^2 m_{D+1} \\ m_3+\sigma^2 m_2 & m_4+\sigma^2 m_3 & \dots & m_{D+3}+\sigma^2 m_{D+2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{D+2}+\sigma^2 m_{D+1} & m_{D+3}+\sigma^2 m_{D+2} & \dots & m_{2D+2}+\sigma^2 m_{2D+1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_D \end{bmatrix}$$

with the moments

$$m_k \triangleq \mathrm{E}\left\{\lambda^k\right\} = \phi\left(\boldsymbol{C}^k\right)$$

The Calculation of Moments

The Z–transform of the moments relates to the Stieltjes transform as

$$\sum_{k=0}^{\infty} m_k s^{-k} = -sG(-s).$$

This yields

$$m_k = P^k \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \binom{k}{i} \binom{k}{j} \binom{k}{i+j+1} \frac{\beta^i \zeta^j}{k} \qquad \forall\, k > 0.$$

Optimum Weights

$D = 1$:

$$
\begin{aligned}
w_1 &= -\beta\zeta - \beta - \zeta \\
w_0 &= -(\sigma^2/P + 2 + 2\beta + 2\zeta)w_1 + \beta\zeta
\end{aligned}
$$

$D = 2$:

$$
\begin{aligned}
w_2 &= \beta^3\zeta^3 + 3\beta^3\zeta + 3\beta^2\zeta + 3\beta\zeta^3 + 3\beta\zeta^2 + 5\beta^2\zeta^2 + 3\beta^3\zeta^2 + 3\beta^2\zeta^3 + \zeta^3 + \beta^3 \\
w_1 &= -(\sigma^2/P + 3 + 3\beta + 3\zeta)w_2 - 2\beta^3\zeta - 4\beta^2\zeta^2 - 2\beta\zeta^3 - 4\beta^3\zeta^2 - 4\beta^2\zeta^3 - 2\beta^3\zeta^3 \\
w_0 &= -\sigma^2/Pw_1 + (3 + 5\beta + 5\zeta + 5\beta\zeta + 3\beta^2 + 3\zeta^2)w_2 + \\
&\quad + 2\beta^2\zeta^2 + 2\beta^4\zeta^2 + 2\beta^2\zeta^4 + 11\beta^3\zeta^3 + 11\beta^3\zeta^2 + 11\beta^2\zeta^3 - \beta^4 - \zeta^4 - \beta^4\zeta^4
\end{aligned}
$$

Easy to calculate in real–time.

Conclusion

The more antennas

- *the higher capacity and*
- *the easier to equalize!*