On Channel Capacity, Uncoded Error Probability, ML-Detection and Spin Glasses

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Abstract— **The paper gives a survey of recent developments in the analysis of large size code-division multiple-access communication systems which indicate that there might be a fundamental and simple relationship between the uncoded probability of error and the channel capacity of such channels.**

I. Introduction

It is well-known that there is an obvious relationship between the uncoded probability of error and channel capacity for some channels. One of these channels is the binary symmetric channel where channel capacity is given by

$$C_{\mathrm{BSC}} = 1 - \mathrm{e}_2(P_\mathrm{e}) \tag{1}$$

with $\mathrm{e}_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ denoting the binary entropy function and P_e denoting the uncoded error probability.

The goal of this paper is to extend such relationship to channels where the connection between uncoded error probability and channel capacity is less obvious. A first step into that direction is the following lemma proven in [1]:

Lemma 1: Consider a memoryless weakly symmetric[1] *channel with binary input and output $Y \in \mathcal{Y}$. Let $P_\mathrm{e}(y)$ denote the minimum uncoded probability of detection error of this channel conditioned on the observation of the output symbol $Y = y$. Then, the capacity of that channel is given by*

$$C = 1 - \mathop{\mathrm{E}}_{y} \mathrm{e}_2\big(P_\mathrm{e}(y)\big) \tag{2}$$

where the minimization of the uncoded probability of error is with respect to the optimum detection rule.

On the one hand, Lemma 1 is more general than (1). On the other hand, it involves the conditional uncoded probability of error.

Note that (1) could be written as

$$C_{\mathrm{BSC}} = 1 - \mathrm{e}_2\left(\mathop{\mathrm{E}}_{y} P_\mathrm{e}(y)\right). \tag{3}$$

[1] See [2] for definition of weak symmetry.

Thus, a potential loss due to binary quantization of the output alphabet can be understood as the loss due to averaging over the binary entropy function.

II. Multiuser Efficiency

One step further into the direction of (2), is a result found recently by Shamai and Verdú [3] who could characterize the capacity of a certain multiuser channel in terms of something called *multiuser efficiency.*

Multiuser efficiency is defined as the loss in power efficiency for uncoded transmission due to the presence of multiuser interference in comparison to the case where only a single user is active on the channel. Thus, in a bit error rate plot, it is the shift between the curve for interfered transmission in comparison to interference-free transmission, as illustrated in Fig. 1.

The multiuser efficiency is a number within the interval $[0;1]$. A multiuser efficiency of $\eta = 1$ (0 dB) means that there is no loss due to multiuser interference. Multiuser efficiency is a performance measure that is by definition directly related to uncoded bit error probability.

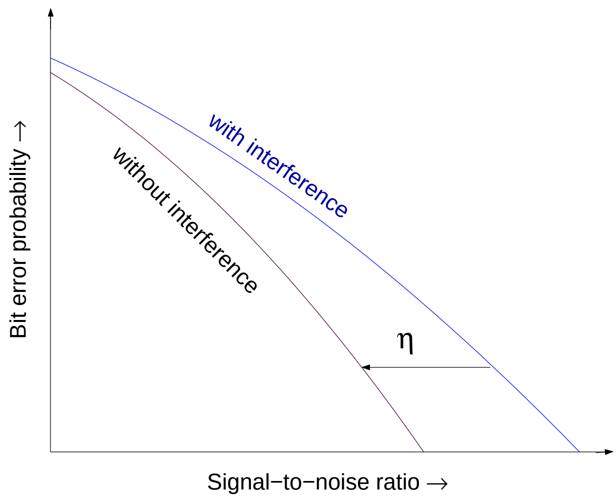

Fig. 1. Multiuser efficiency.

Multiuser efficiency does not only depend on the channel characteristics, but also on the detector that is used to estimate which symbols were transmitted.

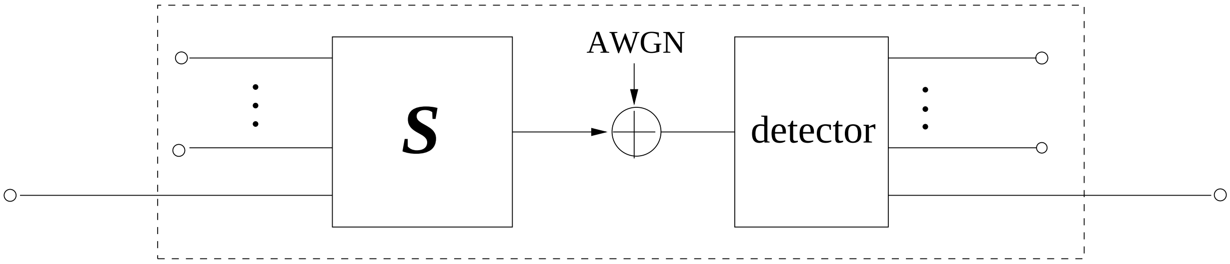

Fig. 2. Super-channel for the Gaussian CDMA channel.

III. Random CDMA

Consider the following multiuser channel

$$\boldsymbol{y}[\mu] = \boldsymbol{S}\boldsymbol{x}[\mu] + \boldsymbol{n}[\mu], \tag{4}$$

where μ is discrete time, the $K \times 1$ vector $\boldsymbol{x}[\mu]$ contains the symbols sent by user k in its k^{th} component, $\boldsymbol{n}[\mu]$ is vector-valued additive white jointly Gaussian noise, the $N \times 1$ vector $\boldsymbol{y}[\mu]$ is the channel output, and the $N \times K$ matrix $\boldsymbol{S}$ is fixed and known. The total capacity of that channel is well-known [4] and can be given in closed form as a function of the matrix $\boldsymbol{S}$.

The dependency of total capacity of that channel on the matrix $\boldsymbol{S}$ vanishes, if $\boldsymbol{S}$ is random (but still known at receiver side) with independent, identically distributed entries and K, N tend to infinity with some fixed ratio $\beta = K/N$ which is called the *load* [5]. It is achieved for Gaussian input alphabet and will be denoted by $C_{\text{total}}^{\text{Gauß}}$ (the explicit formula, however, is not relevant for the development of this paper).

IV. Separation of Detection and Decoding

In order to understand the result of Shamai and Verdú [3] consider now the problem of separated detection and decoding. Since the complexity of decoding a multiuser code is, in general, exponential in both the codeword length and the number of users, it can be of practical interest to separate detection and decoding in order to save complexity, though this approach will lead to a penalty in performance, in general.

From an information-theoretic point of view, one can define a super-channel comprising both the actual channel and the detection unit. An example for the channel (4) is depicted in Figure 2. Comparing the capacity of that super-channel (which by the data processing lemma is not greater than the capacity of the actual channel) to the capacity of the actual channel is a reasonable method to quantify the loss due to separation of detection and decoding.

Since it does not make sense to consider the uncoded probability of error for a continuous input alphabet, we re-strict the input alphabet of the super-channel to be binary, though binary input it is not capacity achieving, in general. Moreover, consider a generalized detector which outputs a soft value instead of a binary decision. In that case, an error occurs for binary antipodal signaling, if the output value has different sign as the channel input.

Assume the detector is adjusted to minimize the mean-squared error between its output and the channel input under the assumption that the input alphabet follows a certain distribution which need not be the true binary distribution. Define the capacity of the super-channel in the large system limits with such a detector as C_{sep}^{pdf}. Then, the result of Shamai and Verdú [3] can be stated as

$$C_{\text{total}}^{\text{Gauß}} - C_{\text{sep}}^{\text{Gauß}} = \frac{\eta^{\text{Gauß}} - 1 - \log \eta^{\text{Gauß}}}{2\beta \log 2} \tag{5}$$

where $\eta^{\text{Gauß}}$ is the multiuser efficiency of the detector making the Gaussian assumption on the input alphabet. Since multiuser efficiency measures the uncoded bit error probability, (5) directly relates relative capacity to a relative uncoded bit error probability.

Remark 1: Note that while $C_{\text{total}}^{\text{Gauß}}$ means that the input alphabet is Gaussian, $C_{\text{sep}}^{\text{Gauß}}$ means that the input alphabet is binary, but the detector operates under the hypothesis that the input alphabet is Gaussian.

The particularly surprising fact about (5) is the simplicity of the capacity difference while the capacities are very complicated expressions in terms of two channel parameters load and signal-to-noise ratio. It was tempting to believe that the simplicity of (5) occurs just by chance or due to the Gaussian assumption until Müller and Gerstacker [1] have recently shown the following result:

$$C_{\text{total}}^{\text{binary}} - C_{\text{sep}}^{\text{binary}} = \frac{\eta^{\text{binary}} - 1 - \log \eta^{\text{binary}}}{2\beta \log 2} \tag{6}$$

This complete analogy between (5) and (6) is even more surprising, if one considers that the multiuser detector for the binary input hypothesis is an np-complete algorithm, while it is a linear filter for the Gaussian input hypothesis.

The previous analogy gives naturally rise to the question whether the following conjecture is true:

Conjecture 1: Let pdf denote an arbitrary (hypothesis on a) channel input alphabet with non-vanishing entropy. Then, we have

$$C_{\text{total}}^{pdf} - C_{\text{sep}}^{pdf} = \frac{\eta^{pdf} - 1 - \log \eta^{pdf}}{2\beta \log 2} \tag{7}$$

in the large system limit of the Gaussian CDMA channel defined in (4).

Proving this conjecture (provided it is true) would not only establish a nice closed form result, but a proof is also expected to disclose a much deeper understanding of multiuser detection and channel capacity. It is unlikely that Conjecture 1 can be proven in a similar way as (5) and (6) which were simply found by calculation of all the four capacities involved—a task which has proven to be very hard even for the simple Gaussian and binary cases. However, even coming up with (6) has required a puzzling new view of maximum a-posteriori detection in context of statistical mechanics discovered by Tanaka [6], [7] and strengthened in analytical rigor in [8]. This analogy is briefly outlined in the next section.

V. ML-Detection and Spin Glasses

A spin glass is a ferromagnetic material. Its overall magnetization is determined by the superposition of binary magnetic spins. Since spins influence each other, describing the magnetic behavior of a spin glass is far from straightforward. One popular description of a spin glass is the Sherrington-Kirkpatrick model [9]. It defines the energy function (Hamiltonian) as

$$H = -\sum_{i<j} r_{ij} x_i x_j - h \sum_i x_i \tag{8}$$

where x_i are the binary spins, h is the external magnetic field, and r_{ij} are the spin interactions which are modeled as random variables. For simplicity and analytical tractability, the number of spins is assumed to be infinite.

Consider now a maximum-likelihood detector for the Gaussian CDMA channel (4). Due to Gaussianity of the noise, the object function to minimize is the Euclidean distance

$$||\boldsymbol{y} - \boldsymbol{S}\boldsymbol{x}||. \tag{9}$$

Defining $\boldsymbol{R} = \boldsymbol{S}^\dagger \boldsymbol{S}$ and $\boldsymbol{h} = \boldsymbol{S}^\dagger \boldsymbol{y}$, minimizing the object function with respect to functions of $\boldsymbol{x}$ is equivalent to minimizing

$$\sum_{i<j} r_{ij} x_i x_j - \sum_i h_i x_i. \tag{10}$$

Comparing (10) to (8), the only essential difference between minimizing the energy of a spin glass and maximum-likelihood detection on the Gaussian CDMA channel, is the fact that on the latter the external magnetic field may differ from one spin to the other. Another less obvious difference is the assumption about the statistics of $\boldsymbol{R}$. Assuming that $\boldsymbol{S}$ is a random matrix with independent identically distributed entries, we have $\boldsymbol{R} = \boldsymbol{S} + \boldsymbol{S}^\dagger$ for spin glasses, while $\boldsymbol{R} = \boldsymbol{S}^\dagger \boldsymbol{S}$ for CDMA. This does indeed significantly change the results of the analysis, but allows for the use of the same analytic tools to attack both problems.

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