Statistics and Chip Pulse Design for Efficient Multiuser Detection in Asynchronous CDMA.

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***Abstract*— The design and analysis of multistage detectors with universal weights for *asynchronous* CDMA systems is presented. The use of a front-end that enables joint detection and provides sufficient statistics is proposed. With such a front end the proposed multistage detector has the same complexity order per bit as the matched filter. The proposed approach can also take into account other suboptimum statistics and the non-ideality of the chip pulse waveforms. In such a way, the universal weights can be designed and the performance can be computed for very realistic scenarios.**

I. Introduction

The design of low complexity linear multiuser detectors with random spreading sequences and their large system analysis is mainly focused on synchronous code division multiple access (CDMA) systems [1]–[3]. The synchronous assumption is not realistic in the uplink of CDMA systems. Asynchronous CDMA systems with matched filters at the receivers have been analyzed first in [4]. In [5] the analysis has been extended to CDMA systems with linear minimum mean square error (MMSE) detectors and sinc pulse modulation. In [6], asynchronous systems with symbol asynchronous but chip synchronous signals, i.e. the time delay of the users is a multiple of the the chip interval, are investigated and an efficient implementation of linear multiuser detectors with the same complexity order per bit as the matched filter is proposed for such a scenario. In this work we extend the design of efficient multistage detectors for uplink CDMA in [3], [6] to totally asynchronous CDMA systems. The low complexity of such detectors is obtained thanks to two essential features: (i) the joint processing of the received signals for all K users of interest so that most of computations becomes identical and the complexity drops by a factor of K, (ii) the use of *universal weights* based on the asymptotic self-averaging properties of random matrices instead of tailored weights depending on the transmitted spreading sequences.

For synchronous CDMA systems sufficient statistics for all users can be obtained sampling at the chip rate and time delay common to all users under some restriction on the chip pulse waveforms[1]. A similar approach does not provide sufficient statistics in case of completely asynchronous systems and the choice of sufficient statistics or suboptimal statistics, trading complexity against performance, plays a relevant role on both complexity and performance of multiuser detectors for asynchronous systems. Furthermore, the effects of the bandwidth and chip-pulse shaping on the performance of CDMA systems become also relevant. In fact, in contrast to synchronous systems the performance of linear multiuser detectors for CDMA systems increases with the bandwidth as analytically shown in [7].

[1]The autocorrelation of the chip pulse waveform satisfies the Nyquist criterion. These chip pulse waveforms are also known as square root Nyquist waveforms.

In this work we analyze the impact of different sets of observables on the design and the performance of low complexity multistage detectors and the effects of pulse shaping waveforms on the performance of linear detectors.

A widely used approach to generate accurate but suboptimal statistics process the received signal by a filter matched to the chip waveform. For a given user, a set of observables is obtained by sampling the filter output at the time delay of the user with rate equal to the chip rate independently of frequency bandwidth of the chip waveforms. We refer to this statistics as Statistics A. This discretization scheme provides different observables for each user and does not enable a joint processing of all users. An implementation of multistage detectors with universal weights using such statistics implies a complexity order *per bit* equal to $\mathcal{O}(K^2)$. This approach is still interesting from a complexity point of view if detection of a single user is required. In this case its complexity order *per bit* is still equal to $\mathcal{O}(K^2)$ while the complexity order *per bit* of the linear MMSE detector is $\mathcal{O}(K^3)$.

In systems with bandlimited waveforms sufficient statistics can be obtained by filtering the received signal by a low pass filter with bandwidth $B_{\rm LOW}$ larger than the chip-pulse bandwidth and subsequent sampling at rate $2B_{\rm LOW}$. We refer to these statistics as Statistics B. These observables enable a joint processing of all users without loss of information and multistage detectors with universal weights have a complexity order per bit equal to $\mathcal{O}(rK)$ if r is the oversampling factor.

In this work the design of low complexity detectors for CDMA systems with chip asynchronism and the corresponding analysis are proposed for both discretization schemes. Our analysis enable an analytical computation of the performance degradation due to the use of sub-optimal statistics as function of few system parameters, namely, the signal-to-noise ratio (SNR), the roll-off, the system load, and the statistics of the channel gains and time delays.

CDMA systems using Statistics A show a substantial dependence of the performance on the initial sampling time instant due to the sub-optimality of the statistics. Sampling at the best time instant, i.e. at the time delay of the user of interest, implies a slight performance degradation at very low SNRs and low roll-off. However, this degradation increases as the SNR and the roll-off increase. Above all in CDMA systems real implementations do not make use of costly pulse shapers. The analysis presented in this work hold for very general chip waveforms not necessarily inter-chip interference free. Therefore, the degradation effects due to the non-ideality of pulse shapers can be computed.

Similarly, the design of universal weights for the multistage detectors proposed in this work takes into account the sub-optimality of statistics or the non-ideality of pulse shapers.

II. System Model

Let us consider an asynchronous CDMA system with K users in the uplink channel and spreading factor N. The received signal in complex base-band notation is given by

$$y(t) = \sum_{k=1}^{K} a_k s_k(t-\tau_k) + n(t) \qquad t \in (-\infty, +\infty)$$

where a_k is the received signal amplitude of user k and takes into account the transmitted symbol amplitude and the flat fading channel, τ_k is the time delay of user k. Without loss of generality (w.l.g.) we assume $\tau_k \in [0, T_s]$ with T_s the symbol interval and $0 \leq \tau_1 \leq \tau_2 \ldots \leq \tau_K$. Here, $n(t)$ is a zero mean complex Gaussian circularly symmetric process with two-sided power spectral density N_0, and $s_k(t)$ is the spread signal of user k,

$$s_k(t) = \sum_{m=-\infty}^{+\infty} b_k[m] c_k^{(m)}(t). \tag{1}$$

In (1), $b_k[m]$ is the transmitted symbol of user k in the symbol interval m and

$$c_k^{(m)}(t) = \sum_{u=0}^{N-1} s_{km}[u] \psi(t - mT_s - uT_c)$$

is its spreading waveform. $s_{km}[u]$, $u = 0, \ldots, N-1$, are elements of the spreading sequence of user k in the m^{th} symbol interval. T_c is the chip interval. The spreading sequences $s_{km}[u]$ are assumed to be i.i.d. random variables with $\mathrm{E}\{s_{km}[u]\} = 0$, $\mathrm{E}\{|s_{km}[u]|^2\} = \frac{1}{N}$, and $\lim_{N\to\infty}\{N^3 |s_{km}[u]|^6\} < +\infty$. The users' symbols $b_k[m]$ are uncorrelated random variables with $\mathrm{E}\{|b_k[m]|^2\} = 1$ and $\mathrm{E}\{b_k[m]\} = 0$. We denote by $\psi(t)$ the band limited chip waveform with bandwidth B, unit energy, and Fourier transform $\Psi(f)$. Thanks to the statistical properties of the spreading sequence, the average energy of the signature waveforms is also unit.

At the front-end, the base band signal is processed by a filter with impulsive response $g(t)$ and corresponding frequency response $G(f)$. We denote by $\phi(t)$ the output of the filter corresponding to the input $\psi(t)$, i.e. $\phi(t) = g(t) * \psi(t)$ and by $\Phi(f)$ its Fourier transform. The filter output is sampled at rate $\frac{r}{T_c}$.

The discrete signal at the front-end output is given by

$$y[p] = \sum_{k=1}^{K} a_k \sum_{m=-\infty}^{+\infty} b_k[m] \sum_{u=0}^{N-1} s_{k,m}[u] \phi\left(\left(\frac{p}{r} - u - mN\right) T_c - \tau_k\right) + n[p]$$

with $p \in \mathbb{Z}$ and $n[t]$ the discrete time, complex-valued noise.

Throughout this work we assume that the filtered chip pulse waveform $\phi(t)$ is much shorter than the symbol waveform, i.e. $\phi(t)$ becomes negligible for $|t| > t_0$ and $t_0 \ll T_s$. This is usually verified in the systems with large spreading factor, which we are considering. Thus, we can neglect the useful signal outside the symbol interval $[0, T_s]$.

Given the time delay τ_k the virtual spreading sequence of user k for the transmitted symbol m spans the symbol intervals m and $m+1$ and it is a $2Nr$-dimensional vector given by

$$\boldsymbol{v}_{km} = \boldsymbol{\Phi}_k \boldsymbol{s}_{km}$$

where $\boldsymbol{s}_{km} = (s_{km}[0] \ldots s_{km}[N-1])^T$ and $\boldsymbol{\Phi}_k$ is a $2Nr \times N$ matrix taking into account the effects of the pulse shape and the time delay of user k. Its (i,j)-element is equal to zero for $i \leq \lfloor \frac{r\tau_k}{T_c} \rfloor$ and for $i \geq \lfloor \frac{r\tau_k}{T_c} \rfloor + Nr$, while $(\boldsymbol{\Phi}_k)_{ij} = \phi\left(\frac{(i-1)T_c}{r} - (j-1)T_c - \tau_k + \lfloor \frac{r\tau_k}{T_c} \rfloor\right)$ for $\lfloor \frac{r\tau_k}{T_c} \rfloor + 1 \leq i \leq \lfloor \frac{r\tau_k}{T_c} \rfloor + Nr$. The zero elements are due to the fact that we neglect the useful signal outside the interval $[mT_s + \tau_k, (m+1)T_s + \tau_k]$.

Let $\boldsymbol{S}[m]$ be the $2rN \times K$ matrix of virtual spreading, i.e. $\boldsymbol{S}[m] = (\boldsymbol{\Phi}_1 \boldsymbol{s}_{1m}, \boldsymbol{\Phi}_2 \boldsymbol{s}_{2m}, \ldots \boldsymbol{\Phi}_K \boldsymbol{s}_{Km})$, $\boldsymbol{A}$ the $K \times K$ diagonal matrix of received amplitudes, $\boldsymbol{H}[m] = \boldsymbol{S}[m]\boldsymbol{A}$, and $\boldsymbol{b}[m]$ and $\boldsymbol{y}[m]$ the vectors of transmitted and received signals, respectively. Furthermore, we decompose the matrix $\boldsymbol{H}[m]$ into two matrices of size $rN \times K$, $\boldsymbol{H}_u[m]$ and $\boldsymbol{H}_d[m]$ such that $\boldsymbol{H}[m] = [\boldsymbol{H}_u^T[m], \boldsymbol{H}_d^T[m]]^T$. Then, the baseband discrete-time asynchronous system in matrix notation is given by

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{N}}$$

where $\boldsymbol{\mathcal{Y}} = [\ldots, \boldsymbol{y}^T(m-1), \boldsymbol{y}^T(m), \boldsymbol{y}^T(m+1) \ldots]^T$ and $\boldsymbol{\mathcal{B}} = [\ldots, \boldsymbol{b}^T(m-1), \boldsymbol{b}^T(m), \boldsymbol{b}^T(m+1) \ldots]^T$ are the infinite-length vectors of received and transmitted symbols respectively; $\boldsymbol{\mathcal{N}}$ is an infinite-length noise vector; and $\boldsymbol{\mathcal{H}}$ is a bi-diagonal block matrix with infinite block rows and block columns given by

$$\boldsymbol{\mathcal{H}} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ldots & \mathbf{0} & \boldsymbol{H}_d(m-1) & \boldsymbol{H}_u(m) & \mathbf{0} & \ldots & \ldots \\ \ldots & \ldots & \mathbf{0} & \boldsymbol{H}_d(m) & \boldsymbol{H}_u(m+1) & \mathbf{0} & \ldots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Additionally, $\boldsymbol{h}_{k,m}$ denotes the column of the matrix $\boldsymbol{\mathcal{H}}$ containing the k^{th} column of the matrix $\boldsymbol{H}[m]$. Here, $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{R}}$ are the correlation matrices defined as $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^H$ and $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{H}}^H\boldsymbol{\mathcal{H}}$. Finally $\beta = \frac{K}{N}$ is the system load.

We focus on the following front-ends.

Front-end Type A consists of an analog filter $G(f)$ matched to the chip pulse, i.e. $G(f) = \Psi^*(f)$, and a subsequent sampler with sampling rate equal to the chip rate. This front-end implies the use of square root Nyquist chip pulse waveforms to keep the discrete noise process white with variance $\frac{N_0}{T_c}$.

Front-end Type B consists of (i) a low pass filter $G(f)$ with low pass band $|f| \leq \frac{r}{2Tc}$ where $r \in \mathbb{Z}^+$ satisfies the constraint $B \leq \frac{r}{2T_c}$ so that the sampling theorem is satisfied and of (ii) a subsequent sampler at rate $\frac{r}{T_c}$. With this choice of the front end the conditions of the sampling theorem are satisfied so that the sampled signal provides sufficient statistics. The sampling rate is a multiple of the chip rate as convenient in practical implementations. Additionally, the discrete noise process $\{n[p]\}$ is still white with zero mean and variance $\sigma^2 = \frac{N_0 r}{T_c}$.

III. Linear Multiuser Detection

The design of multistage detectors with universal weights follows along the design of multistage detectors for symbol asynchronous but chip synchronous CDMA systems in [6]. Multistage detectors of rank $L \in \mathbb{Z}^+$ and $L \leq K$ for symbol m of user k performs a projection onto the Krylov subspace $\chi_{L,k,m}(\boldsymbol{\mathcal{H}}) = \mathrm{span}(\boldsymbol{\mathcal{T}}^\ell \boldsymbol{h}_{k,m})|_{\ell=0}^{L-1}$, for $k = 1...K$, $m \in \mathbb{Z}$ and a subsequent filtering. They are defined as

$$\widehat{b}_k[m] = \sum_{\ell=0}^{L-1} w_{k,\ell} \boldsymbol{h}_{k,\ell}^H \boldsymbol{\mathcal{T}}^\ell \boldsymbol{\mathcal{Y}}$$

where $w_{k,m}$ are the filter coefficients for user k. The multistage detector processing jointly all users is given by

$$\widehat{\boldsymbol{b}}_k[m] = \sum_{\ell=0}^{L-1} \boldsymbol{W}_\ell \boldsymbol{\mathcal{H}}_m^H \boldsymbol{\mathcal{T}}^\ell \boldsymbol{\mathcal{Y}}$$

where $\boldsymbol{\mathcal{H}}_m = (\boldsymbol{h}_{1,m}, \boldsymbol{h}_{2,m} ... \boldsymbol{h}_{K,m})$ and $\boldsymbol{W}_\ell$ is a diagonal matrix with the k^{th} element equal to $w_{k\ell}$. Figure 1 shows an implementation that enables detection with finite delay. The family of multistage detectors includes many well known detectors like multistage Wiener filters, polynomial expansion detectors, or, equivalently conjugate gradient methods, and the parallel interference canceller, eventually weighted. In [6] it is shown that the implementation of multistage Wiener filters in Figure 1 for asynchronous systems with a sufficiently large number of stages can even outperform a linear MMSE detector with finite observation window. Therefore, we focus on the design of a detector with universal weights equivalent to the multistage Wiener filter for large systems (as $K, N \to \infty$). We refer to it as detector Type J-I to emphasize the fact that the detector performs the projection of all users *jointly* and the filter is designed *individually* for each user. The universal weights are obtained as

$$w_{k,m} = \lim_{\substack{N,K\to\infty \\ \frac{K}{N}\to\beta}} w_{k,m}(N)$$

where $w_{k,m}(N)$ are the tailored filter coefficients of the multistage Wiener filter minimizing the MSE $\mathrm{E}\{\|b_k[m] - \sum_{\ell=0}^{L-1} w_{k,m,\ell}(N) \boldsymbol{h}_{k,m}{}^H \boldsymbol{\mathcal{T}}^m \boldsymbol{\mathcal{Y}}\|^2\}$. The tailored weight $w_{k,m,\ell}(N)$ is the $(\ell+1)^{st}$ element of the vector $\boldsymbol{w}_{k,m}(N)$ given by

$$\boldsymbol{w}_{k,m}(N) = \boldsymbol{\Xi}_{k,m}^{-1}(N) \boldsymbol{\xi}_{k,m}(N)$$

where $\boldsymbol{\Xi}_{k,m}(N) = \left((\boldsymbol{\mathcal{R}}^{i+j}(m))_{kk} + \sigma^2 (\boldsymbol{\mathcal{R}}^{i+j-1}(m))_{kk} \right)$, with $i,j = 1 \ldots L$, $\boldsymbol{\xi}_{k,m}(N) = \left((\boldsymbol{\mathcal{R}}^j(m))_{kk} \right)_{j=1\ldots L}$, and $(\boldsymbol{\mathcal{R}}^j(m))_{kk} = \boldsymbol{h}_{k,m}^H \boldsymbol{\mathcal{T}} \boldsymbol{h}_{k,m}$ denote the diagonal element of the matrix $\boldsymbol{\mathcal{R}}^j$ corresponding to user k at time instant m. The design of universal weights reduces to the computation of the asymptotic values $\lim_{K=\beta N\to\infty} (\boldsymbol{\mathcal{R}}^j(m))_{kk}$. The convergence of the diagonal elements of $\boldsymbol{R}^j$ to deterministic values is established in the following theorem.

Theorem 1: Given the Fourier transform $\Phi(j2\pi f)$ with finite support and bounded in absolute value, let $\widetilde{\Phi}_\tau(x) \triangleq \frac{1}{T_c} \sum_{s=-\infty}^{+\infty} \mathrm{e}^{-j2\pi \frac{\tau}{T_c}(x+s)} \Phi^* \left(\frac{j2\pi}{T_c}(x+s) \right)$. Assume that the sequence of the empirical joint distributions $F_{|\boldsymbol{A}|^2,T}^{(K)}(\lambda,\tau) = \frac{1}{K}\sum_{k=1}^K 1(\lambda - |a_{kk}|^2) 1(\tau - \tau_k)$ converges almost surely, as $K \to \infty$, to a non-random distribution function $F_{|\boldsymbol{A}|^2,T}(\lambda,\tau)$ with bounded support[2]. Furthermore, the spectral radius of the matrix $\boldsymbol{\mathcal{R}}$ is almost surely uniformly upper bounded[3]. Then, given $(|a_k|^2, \tau_k)$, the diagonal element $(\boldsymbol{\mathcal{R}}^\ell(m))_{kk}$ of the matrix $\boldsymbol{\mathcal{R}}^\ell$ converges with probability one to a deterministic value, conditionally on $(|a_k|^2, \tau_k)$,

$$\lim_{K=\beta N\to\infty} (\boldsymbol{\mathcal{R}}^\ell(m))_{kk} \overset{a.s.}{=} \mathcal{R}_\ell(|a_{kk}|^2, \tau_k)$$

with $\mathcal{R}_{(\ell)}(|a_{kk}|^2, \tau_k)$ determined by the following recursion

$$\mathcal{R}_\ell(\lambda,\tau) = \sum_{s=0}^{\ell-1} g(\boldsymbol{U}_{\ell-s-1}, \lambda, \tau) \mathcal{R}_s(\lambda,\tau)$$

and, for $0 \leq x \leq 1$,

$$\boldsymbol{U}_\ell(x) = \sum_{s=0}^{\ell-1} \boldsymbol{f}(\mathcal{R}_{\ell-s-1}, x) \boldsymbol{U}_s(x)$$

$$\boldsymbol{f}(\mathcal{R}_s, x) = \beta \int \lambda \boldsymbol{Q}_\tau(x) \mathcal{R}_s(\lambda,\tau) \mathrm{d}\, F_{|\boldsymbol{A}|^2,T}(\lambda,\tau)$$

$$g(\boldsymbol{U}_s, \lambda, \tau) = \lambda \int_0^1 \mathrm{trace}(\boldsymbol{U}_s(x) \boldsymbol{Q}_\tau(x)) \mathrm{d}\, x$$

where $\boldsymbol{Q}_\tau(x) = \boldsymbol{\Delta}_\tau(x) \boldsymbol{\Delta}_\tau^H(x)$ and

$$\boldsymbol{\Delta}_\tau(x) = \left(\widetilde{\Phi}_\tau(x), \widetilde{\Phi}_{\tau - \frac{T_c}{r}}(x) \ldots \widetilde{\Phi}_{\tau - \frac{T_c(r-1)}{r}}(x) \right)^T.$$

The recursion is initialized by setting $\boldsymbol{U}^0(x) = \boldsymbol{I}_r$ and $R_0(\lambda,\tau) = 1$.

Due to lack of space the proof of this theorem is omitted. The interested reader can refer to [8].

The signal-to-interference and noise ratio (SINR) of a detector Type J-I for large CDMA systems is given by

$$\mathrm{SINR}_{JI,k} = \frac{\boldsymbol{\xi}_{k,\infty}^T \boldsymbol{\Xi}_{k,\infty}^{-1} \boldsymbol{\xi}_{k,\infty}}{1 - \boldsymbol{\xi}_{k,\infty}^T \boldsymbol{\Xi}_{k,\infty}^{-1} \boldsymbol{\xi}_{k,\infty}}. \tag{2}$$

where $\boldsymbol{\xi}_{k,\infty} = \lim_{K=\beta N\to\infty} \boldsymbol{\xi}_{k,m}(N)$ and $\boldsymbol{\Xi}_{k,\infty} = \lim_{K=\beta N\to\infty} \boldsymbol{\Xi}_{k,m}(N)$. Given any multistage detector for user k with filter coefficients $\boldsymbol{w}_k$ it holds

$$\mathrm{SINR} = \frac{\|\boldsymbol{w}_k^H \boldsymbol{\xi}_{k,\infty}\|^2}{\boldsymbol{w}_k^H (\boldsymbol{\Xi}_{k,\infty}^{-1} - \boldsymbol{\xi}_{k,\infty} \boldsymbol{\xi}_{k,\infty}^T) \boldsymbol{w}_k}.$$

IV. Effects of Statistics and of Chip Pulse Waveforms

Let us consider a system with front-end A and let us assume that the chip pulse waveform is a square root raised cosine with roll-off $\gamma \in [0,1]$, i.e.

$$\psi(t) = \frac{4\gamma(\frac{t}{T_c})\cos(\pi(1+\gamma)\frac{t}{T_c}) + \sin(\pi(1-\gamma)\frac{t}{T_c})}{\pi t (1 - (4\gamma\frac{t}{T_c})^2)} \quad \gamma \in [0,1].$$

[2] $1(\cdot)$ denotes the indicator function.

[3] This condition can be replaced by the less restrictive condition that the integer positive eigenvalue moments are upper bounded.

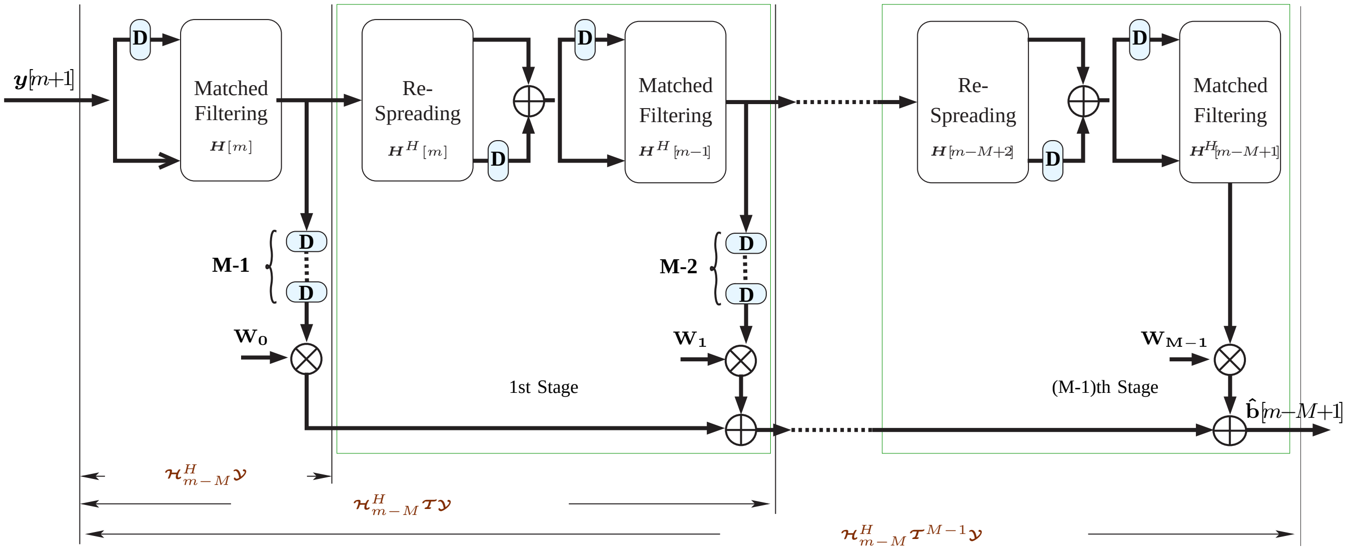

Fig. 1: Multistage detector for asynchronous systems

The limit values $\mathcal{R}_\ell(|a_{kk}|^2, \tau_k)$ can be obtained applying the recursion in Theorem 1 for $r = 1$ and $\boldsymbol{Q}_\tau(x)$ reduced to the scalar

$$\boldsymbol{Q}_\tau(x)=\begin{cases} \frac{1}{2}+\frac{1}{2}\sin^2\left(\frac{\pi}{\gamma}(x+\frac{1}{2})\right)+\frac{\cos 2\pi\tau}{2}\left(1-\sin^2\left(\frac{\pi}{\gamma}(x+\frac{1}{2})\right)\right) & -\frac{1}{2}\le x\le -\frac{1-\gamma}{2}\\ 1 & -\frac{1-\gamma}{2}\le x\le \frac{1-\gamma}{2}\\ \frac{1}{2}+\frac{1}{2}\sin^2\left(\frac{\pi}{\gamma}(x-\frac{1}{2})\right)+\frac{\cos 2\pi\tau}{2}\left(1-\sin^2\left(\frac{\pi}{\gamma}(x-\frac{1}{2})\right)\right) & \frac{1-\gamma}{2}\le x\le \frac{1}{2} \end{cases}.$$

As the front-end B is in use at the receiver, the chip-pulse waveform has bandwidth $\frac{n}{2T_c} \le B \le \frac{n}{T_c}$, the sampling rate is $\frac{r}{T_c}$ with $r \ge n$, and the time delay distribution is uniform, Theorem 1 yields the following algorithm to compute $\mathcal{R}_\ell(|a_{kk}|^2, \tau_k)$.

Algorithm 1:

Initialization

Let $\rho_0(z) = 1$ and $\mu_0(y) = 1$.

ℓ^{th} step

- Define $u_{\ell-1}(y) = y\mu_{\ell-1}(y)$ and write it as a polynomial in y.
- Define $v_{\ell-1}(z) = z\rho_{\ell-1}(z)$ and write it as a polynomial in z.
- Define

$$\mathcal{E}_s = \left(\frac{r}{T_c^2}\right)^s \int_{-\frac{n}{2}}^{\frac{n}{2}} \Phi^{2s}\left(j\frac{2\pi}{T_c}x\right) \mathrm{d}\, x \tag{3}$$

and replace all monomials $y, y^2, \ldots, y^\ell$ in the polynomial $u_{\ell-1}(y)$ by $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_\ell$, respectively. Assign the result to $U_{\ell-1}$.
- Define $m^{(s)}_{|\boldsymbol{A}|^2} = \mathrm{E}\{|a_k|^{2s}\}$ and replace all monomials $z, z^2, \ldots, z^\ell$ in the polynomial $v_{\ell-1}(z)$ by the moments $m^{(1)}_{|\boldsymbol{A}|^2}, \ldots m^{(\ell)}_{|\boldsymbol{A}|^2}$, respectively, Assign the result to $V_{\ell-1}$.
- Set

$$\rho_\ell(z) = \sum_{s=0}^{\ell-1} zU_{\ell-s-1}\rho_s(z)$$

$$\mu_\ell(y) = \sum_{s=0}^{\ell-1} \beta y V_{\ell-s-1}\mu_s(y).$$

- Assign $\rho_\ell(\lambda)$ to $R_\ell(\lambda)$.

Replace all monomials $z, z^2, \ldots, z^\ell$ in the polynomial $\rho_\ell(z)$ by the moments $m^{(1)}_{|\boldsymbol{A}|^2}, \ldots, m^{(\ell)}_{|\boldsymbol{A}|^2}$, respectively, and assign the result to $m^{(\ell)}_{\boldsymbol{\mathcal{R}}}$, the ℓ^{th} eigenvalue moment of order ℓ of $\boldsymbol{\mathcal{R}}$. Algorithm 1 is derived in [8].

Interestingly, $\mathcal{R}_\ell(|a_{kk}|^2, \tau_k)$ is independent of the arrival time in this case and, thus, also the performance of such a CDMA system.

In Figure 2 we compare the performance of a detector Type J-I processing the observables provided by front-end B (dashed lines) to the performance of the corresponding Type J-I detector fed by front-end A (dots) assuming equal received powers, system load $\beta = \frac{1}{2}$, $L = 3$. The chip waveform is a square root raised cosine. The curves represent the output SINR as a function of the roll-off γ parameterized with respect to $\frac{E_s}{N_0}$. The parameter $\frac{E_s}{N_0}$ varies from 0 dB to 20 dB in steps of 5 dB. As reference we also plot the performance of synchronous CDMA systems. As expected, multistage detectors with front-end B outperform the corresponding multistage detectors with front-end A. Interestingly, in the former case linear multiuser detection and asynchronism can compensate to some extent for the loss in spectral efficiency caused by the increasing roll-off and typical of synchronous CDMA systems (note that the SINR keeps constant for synchronous CDMA systems) [7]. On the contrary, even in the case of asynchronism, multiuser detection with front-end A can not compensate for the loss in spectral efficiency due to non-zero roll-off and its performance is close to the performance of synchronous CDMA systems for any SNR level.

We elaborate on the effects of the chip pulse waveforms by focusing on square root raised cosine and raised cosine chip-pulse waveforms with roll-off $\gamma \in [0, 1]$. To simplify the notation we assume $T_c = 1$. Let

$$\Upsilon(x) = \begin{cases} 1 & 0 \le |x| \le \frac{1-\gamma}{2}\\ \frac{1}{2}\left(1 - \sin\frac{\pi}{\gamma}\left(|x| - \frac{1}{2}\right)\right) & \frac{1-\gamma}{2} \le |x| \le \frac{1+\gamma}{2}\\ 0 & |x| \ge \frac{1+\gamma}{2}. \end{cases}$$

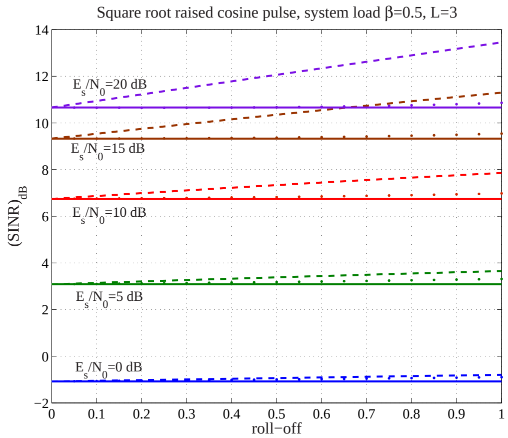

Fig. 2: Asymptotic output SINR of a Type J-I detector with $L = 3$ versus the roll-off γ as front-end A (dashed lines) and front-end B (dots) are in use in an asynchronous CDMA system. The solid lines show the reference performance in synchronous CDMA systems. The curves are parametric in $\frac{E_s}{N_0}$ with $\frac{E_s}{N_0}$ varying between 0 dB and 20 dB in steps of 5 dB.

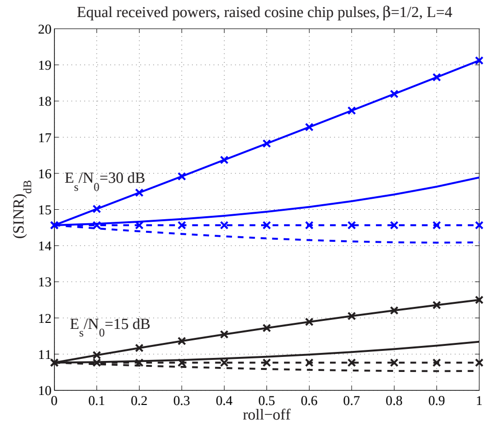

Fig. 3: Asymptotic output SINR of a Type J-I detector with $L = 4$ versus the roll-off γ for square root raised cosine (lines with markers) and raised cosine (lines without markers) chip-pulse waveforms. Solid lines correspond to asynchronous systems while dashed lines refer to synchronous CDMA.

The energy frequency spectrum of a square root raised cosine waveform with unit energy is given by $|\Psi_{\rm sqrc}(j2\pi x)|^2 = \Upsilon(x)$. The unitary Fourier transform of a raised cosine chip waveform with unit energy is $\Psi_{\rm rc}(j2\pi x) = \frac{4}{\sqrt{4-\gamma}}\Upsilon(x)$. We apply (2) and Algorithm 1 to derive the asymptotic SINR to the case of synchronous systems and uniformly distributed time delays. Figure 3 shows the large system output SINR of detectors Type J-I with $L = 4$ versus the roll-off for $\frac{E_s}{N_0} = 15$ dB and $\frac{E_s}{N_0} = 30$ dB, when square root raised cosine (marked lines) and raised cosine (lines with markers) chip-pulse waveforms are transmitted. The dashed lines correspond to synchronous systems while the solid lines refer to asynchronous systems. The SINR is obtained assuming equal received powers at the receiver and system load $\beta = \frac{1}{2}$. Increasing the roll-off the performance increases for both waveforms. At high roll-off the gap in performance between the two different chip-pulse waveforms is quite relevant in favor of the square root raised cosine pulses. Therefore, the design of pulse shapes has a relevant impact. The asynchronous systems outperform the synchronous systems, as already shown in [7]. For synchronous systems, an increase of the utilized bandwidth, due to an increase of the roll-off, can even determine a performance loss if the chip pulse in not properly chosen as it occurs for raised cosine waveforms.

V. Conclusions

In this work we provided guidelines for the design of multistage detectors with universal weights in large asynchronous CDMA systems. The universal weights proposed for the design of low complexity detectors can take into account the effects of asynchronism, sub-optimality of the statistics, and non-ideality of pulse-shapers. With the twofold object of using sufficient statistics and performing jointly the detection of all users of interest we propose the use of front-end B.

From the asymptotic analysis and design performed in this work we can draw the following conclusions:

- Multistage detectors with front end A and universal weights are suboptimal and have the same complexity order per bit of the linear MMSE detector in uplink.
- Multistage detectors with front end B and universal weights are asymptotically optimum and have the same complexity order per bit as the matched filter.
- If only a user had to be detected multistage detectors with front end A have slightly lower complexity than multistage detectors with front end B. However, while the optimum one can compensate to some extent the loss in spectral efficiency due to the roll-off in asynchronous systems multistage detectors with front end A do not have this property. They perform almost as the multistage detectors for synchronous systems at any SNR.

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