(IEEE WiOpt, Riva Del Garda, Trentino, Italy, April 2005)

On the Transport Capacity of Gaussian Multiple Access and Broadcast Channels

G. A. Gupta*, S. Toumpis†, J. Sayir†, R. R. Müller‡
* Mathematics Department, Indian Institute of Technology, New Delhi, India, 110016, gupta@ftw.at
† Telecommunications Research Center Vienna (ftw.), Donau-City-Strasse 1/3, A-1220, Austria, {toumpis,sayir}@ftw.at
‡ Norwegian University of Science and Technology, 7491 Trondheim, Norway, mueller@iet.ntnu.no

***Abstract*— We study the transport capacity of a Gaussian multiple access channel, which consists of a set of transmitters and a single receiver. The transport capacity is defined as the sum, over all transmitters, of the product of the transmission rate with a reward $r(x)$, which is a function of the distance x between the transmitter and the receiver, and quantifies the usefulness of transmitting information over a distance x.**

Assuming that the sum of the transmitter powers is upper bounded, we present in closed form the optimal power allocation among the transmitters, that maximizes the transport capacity. We then present simple expressions for the optimal power allocation and induced transport capacity, as the number of transmitters approaches infinity.

We also study the transport capacity of a Gaussian broadcast channel, which consists of a single transmitter and multiple receivers. Here, the transport capacity is defined as the sum, over all *receivers*, of the product of the transmission rate with a reward $r(x)$. We determine in closed form the maximum possible transport capacity and the distribution of the available transmitter power among the receivers that achieves it. Although this result has already been reported in the literature, our derivation is shorter, and leads to simpler expressions.

Our results can be used to gain intuition and develop good design principles in a variety of settings. For example, they apply to the uplink and downlink channel of cellular networks, and also to sensor networks which consist of multiple sensors that communicate with a single central station.

I. INTRODUCTION

A. Transport Capacity

Consider a wireless multihop network in which a particular node T scheduled to transmit has two options: either transmit to a destination node D_1 with rate R_1, or transmit to a destination node D_2 with rate $R_2 < R_1$. Assume that both transmissions will require the same amount of bandwidth and power, and will convey information of equal importance. In this setting, which destination should node T prefer? Traditional thinking suggests that T should transmit to the node to which it can send data with the highest rate, i.e., D_1. However, in a *multihop* wireless network, in which every packet will have to be transmitted multiple times to reach its final destination, it is not only important that a node transmits *with high rate*, but also that the signal will travel *a large distance*. Indeed, the smaller the distance that a transmission covers, the higher is the number of transmissions of a similar type that are needed, before the transmitted packet reaches its final destination.

Taking this argument a step further, we can argue that a natural figure of merit for the usefulness of a transmission is neither its rate nor the distance covered, but rather the *product* of the two, measured in $\mathrm{bps} \times \mathrm{m}$. Indeed, if two transmissions have the same rate-distance product, using either of the two repeatedly to transmit a given volume of data to a distant destination would consume the same power and bandwidth, even if their rates and covered distances differ significantly. It follows that the summation of the rate-distance products over all transmissions that are active at a given time instant in a wireless network, termed the **transport capacity**, is a natural figure of merit about how efficiently the network operates at that particular instant.

B. Related Work

The obvious relevance of transport capacity in wireless multihop networks is coupled with a remarkable amenability to analysis, as has been demonstrated recently by a number of studies with disparate approaches, a few of which we now briefly review.

Transport capacity was first defined in [1]. There, the authors consider a wireless network of n nodes, placed in a bounded two-dimensional region. It is assumed that the power of transmitted signals decays with distance according to a power law, and that a signal is successfully received if the Signal to Interference and Noise Ratio (SINR) at the receiver is above a fixed threshold. All transmissions are with a fixed global rate W. It is shown that the transport capacity under *any* placement of nodes will have to be smaller than $k_1\sqrt{n}$, where k_1 is a constant independent of n. The bound comes from the fact that any transmission invariably creates interference to other transmissions near by. On the other hand, the authors give examples of network topologies that can sustain a transport capacity greater than $k_2\sqrt{n}$, where $k_2 < k_1$ is another constant, also independent of n.

In [2], the authors consider a probabilistic setting in which nodes are placed on a two-dimensional plane according to a Poisson process with fixed density λ and access a common wireless channel through Aloha. As in [1], transmissions are with a fixed global rate, reception is successful as long as the SINR is above a given threshold, and signal power decays with distance according to a power law. In such an environment,

Work supported by **K***plus* funding for the ftw. project I0 “Signal and Information Processing,” and the European Network of Excellence NEWCOM. The corresponding author is S. Toumpis.

transmitting over longer distances is more useful to the nodes, but on the other hand the probability of successful reception diminishes, as the power of a signal decays with distance and the signal becomes more susceptible to interference from competing transmissions. Therefore, there is a critical tradeoff between the probability of successful reception and the usefulness of the transmission. The authors calculate the optimal distance that signals should attempt to cover, that maximizes the expected transport capacity of the network, and show that under this optimal behavior, the transport capacity can be on the order of $\sqrt{\lambda}$, where λ is the density of nodes. As the density λ is proportional to the expected number of nodes n at a given area, this result is reminiscent of the result in [1].

In [3], the usefulness of a link is described in terms of the product of the communication rate with a reward $r(x)$, where x is the distance between the transmitter and the receiver, and the *reward function* $r(\cdot)$ quantifies the usefulness of transmitting a bit of information over a distance x. In the special case where $r(x) = x$, we get the standard rate-distance product, however the more general case allows the study of alternative notions of usefulness. The authors study a Gaussian broadcast channel consisting of a single transmitter and multiple receivers, placed at increasing distances from the transmitter. The propagation environment is described in terms of the noise-to-signal (NSR) function $s(\cdot)$, defined so that the noise to signal ratio experienced by a receiver at a distance x from the transmitter is equal to $s(x)$. The capacity region of this network, i.e., the sets of simultaneously achievable rates of communication from the transmitter to each of the receivers, is known [4]. The authors build on this knowledge to calculate the point in the capacity region that maximizes the transport capacity of the network, defined here as the summation, over all receivers, of their respective rate-reward products. Under a fairly general assumption on the relation of the reward and NSR functions, the authors are able to find closed form expressions for the maximum transport capacity.

In [5] we continue along the information theoretic tangent initiated in [3], using the notion of transport capacity defined there. In particular, we consider the Gaussian multiple access channel of Fig. 1 that consists of a single receiver and n transmitters $T_1,\ T_2, \ldots,\ T_n$, placed at increasing distances $x_1, x_2, \ldots, x_n$, from the receiver. For each transmitter T_i, there is a maximum power P_i with which it can transmit. The capacity region of this network is also known [4]. In particular, points on the boundary of the capacity region can be achieved if all the transmitters transmit simultaneously to the receiver, with their full power, and the receiver successively decodes the incoming signals. Depending on the decoding order, different points on the boundary of the capacity region are achieved, corresponding to different values of the transport capacity. We show that, as long as the reward function is increasing with distance, the decoding order that maximizes the transport capacity is the one that starts from signals coming from nearby transmitters, and moving outwards. In addition, if a bound is placed on the sum of transmitter powers (but not on the power of individual transmitters), the maximization of the transport capacity can be formulated as a convex optimization problem.

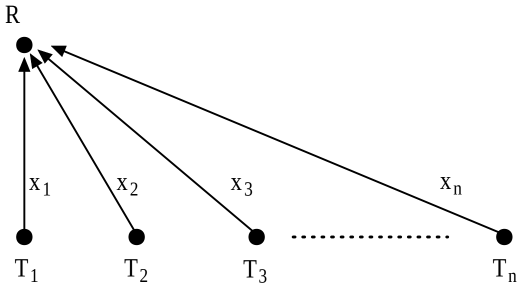

Fig. 1. The multiple access channel, which consists of a single receiver R and n transmitters T_i, with $i = 1, 2, \ldots, n$, placed at increasing distances x_i from the receiver.

C. Contributions

Here, we conclude the investigation that we initiated in [5], by deriving a closed form solution for the optimal allocation of powers in multiple access channel, under the sum-power constraint, and under a fairly general assumption on the relation between the reward and decay functions. The closed form solution is simple enough to allow the calculation of the limiting power allocation, as the number of nodes goes to infinity.

We also calculate the closed form solution for the optimal allocation of powers in the Gaussian broadcast channel studied in [3]. Although this result has already been derived in [3], our derivation uses standard tools (the Karush-Kuhn-Tucker conditions), is shorter in length, and leads to a simplified closed form solution.

The rest of the parer is organized as follows: In Section II we present the multiple access network model and state our power allocation problem. To keep the paper self-contained, we also briefly mention some of the results of [5]. In Section III we calculate the closed form solution of the optimal power allocation, and in Section IV we derive asymptotic results for the case when the number of nodes approaches infinity. In Section V, we replicate our derivations for the case of the Gaussian broadcast channel. We conclude in Section VI.

II. MULTIPLE ACCESS NETWORK AND PROBLEM STATEMENT

We consider the multiple access network of Fig. 1 that consists of a single receiver R and n transmitters T_i, where $1 \leq i \leq n$, placed at distances x_i from the receiver, with $x_{i+1} > x_i$ and $x_1 > 0$. The total bandwidth available to the network is equal to B. The receiver is susceptible to additive white Gaussian noise (AWGN) with spectral power density η, and each T_i has a maximum power P_i[1].

When T_i transmits with power p, R will receive the signal with power $p \times d(x_i)$, where the **decay function** $d(\cdot)$ captures

---

[1] More formally, T_i can transmit with any power any time it uses the channel, as long as the average power over time converges with probability 1 to a value equal or smaller than P_i. Alternative constraints have been considered in a similar setting in [6], [7].

the decay of the signal power with distance and is positive and strictly decreasing. For compactness, we use the notation $d_i = d(x_i)$. A decay function of particular interest is the **monomial decay function**, defined by $d(x) = Kx^{-\gamma}$, where $\gamma > 0$ is the **decay exponent** and the constant $K > 0$.

The capacity region $\mathcal{C}$ of this network, i.e., the complete set of all combinations of rates $\{R_i\}$ with which the transmitters T_i can send information concurrently to the single receiver, is known [4]. In particular, each of the transmitters T_i can send information to the receiver with rate R_i as long the following equations are satisfied:

$$\sum_{i\in\mathcal{I}} R_i \leq B\log_2(1+\frac{\sum_{i\in\mathcal{I}} d_i P_i}{\eta B}) \quad \forall \mathcal{I} \subseteq \{1,2,\ldots,n\}. \quad (1)$$

Equations (1) show that the capacity region $\mathcal{C}$ is a convex polyhedron in the n-th dimensional Euclidean space. It can be shown that the polyhedron has exactly $n!$ vertices whose n components are all positive [4]. Each of these vertices can be achieved by employing a successive decoding scheme, under which the receiver decodes the signals coming from the transmitters one by one. When decoding the signal coming from a particular transmitter T_i, the decoding process is not affected by signals that have already been decoded, as these are known and can be subtracted from the compound signal. The rest of the signals will act as interference.

We now associate the transmission of a bit of information across a distance x with a reward $r(x)$, where the **reward function** $r(\cdot)$ is strictly increasing, with $r(0) = 0$. For compactness, we use the notation $r_i \triangleq r(x_i)$ and we also define $r_0 \triangleq r(0) = 0$. A reward function of particular interest is the **monomial reward function** $r(x) = x^\rho$ where $\rho > 0$ is the **reward exponent**. The **transport capacity** associated with the point $\{R_i\} \in \mathcal{C}$ is defined as

$$C_T(\{R_i\}) \triangleq \sum_{i=1}^{n} r_i R_i.$$

In the special case of the monomial reward function with $\rho = 1$, this definition coincides with the original definition of transport capacity given in [1].

A problem that naturally arises is the calculation of the maximum value of the transport capacity, and the point in the capacity region in which it is achieved. In other words, we would like to find the following:

$$\max_{\{R_i\}\in\mathcal{C}} C_T(\{R_i\}) = \max_{\{R_i\}\in\mathcal{C}} \sum_{i=1}^{n} r_i R_i.$$

Note that we are interested in the maximization of a linear function over a polyhedron, a closed and convex set. Therefore, we know that the supremum is achieved, and it makes sense to talk about a maximum. In addition, the optimization is a linear program [8]. Furthermore, the following theorem (proved in [5]) shows that the point in the capacity region that maximizes the transport capacity corresponds to a remarkably straightforward decoding order:

Theorem 1: The maximum transport capacity is achieved only by the successive interference cancellation scheme under which all transmitters transmit with their maximum power, and the signal of transmitter T_j is decoded j-th. The maximum transport capacity is equal to:

$$B\sum_{i=1}^{n} r_i \log_2(1+\frac{d_i P_i}{\eta B + \sum_{k=i+1}^{n} d_k P_k}).$$

Until now it was assumed that each transmitter has a maximum power P_i with which it can transmit. A natural extension of our investigation is to assume that transmitters no longer have individual constraints on their transmitted powers, but rather the sum of powers must be smaller than or equal to some global constant P_0. For each distribution of powers whose sum does not exceed P_0, Theorem 1 applies. Therefore, to maximize the transport capacity in this setting, we need to solve the following optimization problem:

$$\begin{array}{ll} \text{maximize:} & B\sum_{i=1}^{n} r_i \log_2(1+\frac{d_i P_i}{\eta B+\sum_{k=i+1}^{n} d_k P_k}), \\ \text{subject to:} & \begin{cases} \sum_{k=1}^{n} P_k \leq P_0, \\ P_i \geq 0,\ i=1,\ldots,n. \end{cases} \end{array} \quad (2)$$

This problem is pertinent in a number of different settings. For example, in the deployment phase of a sensor network consisting of a single central node and many sensors that must forward information to the central node, if our power sources are limited, we would like to know what is their optimal distribution over the sensors that will lead to the most efficient operation of the network, where we quantify efficiency with the notion of transport capacity. As another example, consider a cellular network, in which many mobile stations in a cell want to access the single base station in the cell, and the network has placed an upper bound on the total transmitted power coming from that cell, in order to bound the interference experienced in neighboring cells that share the same frequency band. In such a setting, we would like to determine the maximum possible transport capacity, because this information will suggest how large the cell can be made.

III. OPTIMAL POWER ALLOCATION

A. Basic Properties

We rewrite the optimization problem (2) as:

$$\begin{array}{ll} \text{minimize:} & f_0(P_1,\ldots,P_n) = \\ & -B\sum_{i=1}^{n}(r_i - r_{i-1})\log_2(\eta B + \sum_{k=i}^{n} d_k P_k) \\ & +Br_n \log_2(\eta B), \\ \text{subject to:} & \begin{cases} \sum_{k=1}^{n} P_k \leq P_0, \\ P_i \geq 0,\ i=1,\ldots,n. \end{cases} \end{array} \quad (3)$$

We note that the objective function $f_0(P_1,\ldots,P_n)$ is convex. Indeed, it can be written as the composite function $h(a_1(P_1,\ldots,P_n),\ldots,a_n(P_1,\ldots,P_n))$, where the function $h : (\mathbf{R}^+)^n \to \mathbf{R}$ is defined by $h(a_1,\ldots,a_n) = -B\sum_{i=1}^{n}(r_i - r_{i-1})\log_2 a_i + Br_n\log_2(\eta B)$, and the functions $a_i : (\mathbf{R}^+)^n \to \mathbf{R}^+$ are defined by $a_i \triangleq \eta B + \sum_{k=i}^{n} d_k P_k$, for $i = 1,\ldots,n$. Noting that the sequence $\{r_i\}$ is strictly increasing, we can easily show that the Hessian of the function h is positive definite, therefore h is convex. In addition, h is non-increasing in each argument. The functions $a_i(P_1, \ldots, P_n)$ are linear, and hence concave. It follows that the composition $h(a_1(P_1, \ldots, P_n), \ldots, a_n(P_1, \ldots, P_n))$ is convex [8]. As the two inequality constraints of (3) are linear, it follows that (3) is a convex optimization problem.

The optimization function f_0 is continuous, and the domain of the problem, i.e., the set of power vectors where the constraints are satisfied, is compact. Therefore, the infimum of f_0 is actually achieved, and it makes sense to discuss about a minimum. Furthermore, only one power distribution achieves this minimum. To see why, let us assume that there are actually two power distributions, $\{P_i^1\}$ and $\{P_i^2\}$ achieving it. Because the mapping from the space of $\{P_i\}$ to the space of $\{a_i\}$ is one-on-one, there are two distinct points $\{a_i^1\}$ and $\{a_i^2\}$ where the function $h(\cdot)$ achieves its minimum. However, the Hessian of $h(\cdot)$ is positive definite in $(\mathbf{R}^+)^n$, therefore $h(\cdot)$ is strictly convex and must have a unique minimum. Therefore, we arrive at a contradiction.

To conclude, the objective function f_0 is minimized (and the transport capacity is maximized) for a unique optimal power distribution $\{P_i^*\}$.

B. A Basic Assumption

To derive the optimal power distribution, we will need to make the assumption that the following inequalities hold:

$$\frac{r_i - r_{i-1}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}} \geq \frac{r_{i+1} - r_i}{\frac{1}{d_{i+1}} - \frac{1}{d_i}}, \ \forall i = 1, \ldots, n-1. \tag{4}$$

Note that we have defined $r_0 \triangleq 0$, and now we also define $d_0 \triangleq \infty$. In other words, the sequence $\frac{r_i - r_{i-1}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}}$ is decreasing.

To derive intuition about (4), we rewrite it as follows:

$$\frac{\frac{1}{d_{i+1}} - \frac{1}{d_i}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}} \geq \frac{r_{i+1} - r_i}{r_i - r_{i-1}}, \ \forall i = 1, \ldots, n-1.$$

The inverse of the decay function, which is increasing, equals the factor by which a signal looses its power at some distance. Therefore, our assumption implies that the signals are loosing their power with distance faster than the value we place on their delivery (through the reward function) increases with distance.

This assumption is not as restricting as it would first seem, as it is satisfied in most cases of practical interest. For example, by the following Lemma 1(iii), we see that (4) is satisfied for the case of the monomial decay and reward functions with $\gamma \geq \rho$. For all real life propagation environments that are approximated by the monomial decay function, $\gamma > 2$. In addition, as discussed in the introduction, the most interesting case of the monomial reward function is when $\rho = 1$. Therefore, the condition $\gamma \geq \rho$ is satisfied in almost all cases of interest. It is also interesting to note that a very similar assumption was needed for the derivation of the optimal power distribution of the *broadcast* Gaussian channel in [3] (see also Section V).

Equations (4) have a number of useful properties, that we will use later on. We collect all of them in the following lemma:

Lemma 1:

(i) *Equations (4) are equivalent with the following:*

$$\frac{r_{i-1}d_{i-1} - r_i d_i}{d_{i-1} - d_i} \leq \frac{r_i d_i - r_{i+1} d_{i+1}}{d_i - d_{i+1}}, \ \forall i = 1, \ldots, n-1, \tag{5}$$

$$\frac{\frac{r_{i-1}}{d_i} - \frac{r_i}{d_{i-1}}}{r_i - r_{i-1}} \leq \frac{\frac{r_i}{d_{i+1}} - \frac{r_{i+1}}{d_i}}{r_{i+1} - r_i}, \ \forall i = 1, \ldots, n-1. \tag{6}$$

(ii) *Equations (4) imply that* $\forall i = 1, \ldots, n-1, \ \forall k = 1, \ldots, n-i$,

$$\frac{r_i - r_{i-1}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}} \geq \frac{r_{i+k} - r_{i-1}}{\frac{1}{d_{i+k}} - \frac{1}{d_{i-1}}}. \tag{7}$$

(iii) *Let* $r(\cdot)$ *and* $d(\cdot)$ *be differentiable in* $(0, \infty)$. *Then (4) holds for all transmitter placements* $\{x_i\}$ *if and only if the function* $\frac{(\frac{1}{d})'}{r'}$ *is increasing, i.e.,*

$$y > x > 0 \Rightarrow \frac{(\frac{1}{d})'(y)}{r'(y)} \geq \frac{(\frac{1}{d})'(x)}{r'(x)}.$$

Proof:

(i) To prove the equivalence of (4), (5), and (6), we cross-multiply all of them and note that we arrive at identical inequalities.

(ii) To prove (7), we use induction. In particular, we prove (7) first for $k = 1$ and for all $i = 1, \ldots, n-1$. For this, we note that, from (4), we have that

$$\frac{r_{i+1} - r_i}{r_i - r_{i-1}} \leq \frac{\frac{1}{d_{i+1}} - \frac{1}{d_i}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}}, \quad i = 1, \ldots, n-1,$$

Adding 1 to each size, simplifying and rearranging terms, we arrive at (7) for $k = 1$ and for all $i = 1, \ldots, n-1$. We now make the inductive hypothesis that (7) holds for some $k \in 1, \ldots, n-2$, and for all $i = 1, \ldots, n-k$. We will show that it then holds for $k+1$, and for all $i = 1, \ldots, n-k-1$. For this, we note that

$$\frac{r_i - r_{i-1}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}} \geq \frac{r_{i+1} - r_i}{\frac{1}{d_{i+1}} - \frac{1}{d_i}} \geq \frac{r_{i+k+1} - r_i}{\frac{1}{d_{i+k+1}} - \frac{1}{d_i}}, \ i = 1, \ldots, n-k-1.$$

The first inequality comes from (4), and the second from the induction hypothesis. Therefore,

$$\frac{r_{i+k+1} - r_i}{r_i - r_{i-1}} \leq \frac{\frac{1}{d_{i+k+1}} - \frac{1}{d_i}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}}, \quad i = 1, \ldots, n-k-1.$$

By adding 1 to each side, simplifying and rearranging terms, we arrive at (7) for $k+1$, and for all $i = 1, \ldots, n-k-1$.

(iii) Let us first assume that (4) holds for all transmitter placements. It will then hold for the case of four transmitters placed at locations $x - \epsilon, x, y - \epsilon, y$, where $\epsilon < y - x$. By applying (4) twice, we have:

$$\frac{r(x) - r(x-\epsilon)}{\frac{1}{d(x)} - \frac{1}{d(x-\epsilon)}} \geq \frac{r(y-\epsilon) - r(x)}{\frac{1}{d(y-\epsilon)} - \frac{1}{d(x)}} \geq \frac{r(y) - r(y-\epsilon)}{\frac{1}{d(y)} - \frac{1}{d(y-\epsilon)}}.$$

By dividing the numerators and denominators of the left and right hand side by ϵ, and taking $\epsilon \to 0$, we find that $\frac{(\frac{1}{d})'(y)}{r'(y)} \geq \frac{(\frac{1}{d})'(x)}{r'(x)}$.

The inverse direction follows similarly to Lemma 1 of [3]. □

C. The Closed Form Solution

It follows by the convexity of the problem [8], that, to prove the optimality of a power allocation $\{P_i\}$, it suffices to show that it satisfies the Karush-Kuhn-Tucker (KKT) conditions, which in our problem become:

$$\sum_{k=1}^{n} P_k = P_0,\ P_i \geq 0,\ \lambda_i \geq 0,\ \frac{df_0}{dP_i} - \lambda_i + \nu = 0,\ \lambda_i \cdot P_i = 0,$$

for $i = 1, \ldots, n$ and for some $\nu, \lambda_i \in \mathbf{R}$. This set of equations can easily be shown to be equivalent to the following:

$$\sum_{k=1}^{n} P_k = P_0,\ P_i \geq 0,\ \frac{df_0}{dP_i} + \nu \geq 0,\ P_i \cdot \left[\frac{df_0}{dP_i} + \nu\right] = 0,$$

for $i = 1, \ldots, n$, and for some $\nu \in R$. By substituting the derivatives of f_0, and setting $\lambda \triangleq \frac{\log 2}{B}\nu$, this set of equations becomes

$$\sum_{k=1}^{n} P_k = P_0,\ P_i \geq 0,\ d_i \sum_{k=1}^{i} \frac{r_k - r_{k-1}}{a_k} - \lambda \leq 0,$$
$$P_i \cdot \left[d_i \sum_{k=1}^{i} \frac{r_k - r_{k-1}}{a_k} - \lambda\right] = 0, \tag{8}$$

for $i = 1, \ldots, n$ and some $\lambda \in \mathbf{R}$. Note that we have defined $a_i \triangleq \eta B + \sum_{k=i}^{n} d_k P_k$. Because a_i equals the residual power of all signals that have not been decoded yet (plus the thermal noise), at the beginning of the i-th decoding step, we will call $\{a_i\}$ the **residual power sequence**. We also let $a_{n+1} \triangleq \eta B$, so that $P_i = \frac{a_i - a_{i+1}}{d_i}$ for $i = 1, \ldots, n$.

We now make the claim that the optimal power distribution $\{P_i^*\}$ satisfies the following set of equations:

$$\begin{aligned}
\sum_{k=1}^{n} P_k &= P_0, && (9)\\
P_i &\geq 0, \quad i = 1, \ldots, L, && (10)\\
d_i \sum_{k=1}^{i} \frac{r_k - r_{k-1}}{a_k} - \lambda &= 0, \quad i = 1, \ldots, L, && (11)\\
P_i &= 0, \quad i = L+1, \ldots, n, && (12)\\
d_i \sum_{k=1}^{i} \frac{r_k - r_{k-1}}{a_k} - \lambda &< 0, \quad i = L+1, \ldots, n, && (13)
\end{aligned}$$

for some $\lambda \in \mathbf{R}$ and an index L, which we call the **cutoff index**. If (9)-(13) hold, then (8) will also hold, and $\{P_i\}$ are the unique optimal power allocation. Our strategy therefore will be to find a power distribution $\{P_i\}$ that satisfies (9)-(13).

Equations (10), (11), and (12) will be satisfied if we set:

$$a_i = \begin{cases} \frac{1}{\lambda} \frac{r_i - r_{i-1}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}} & i = 1, \ldots, L, \\ \eta B & i = L+1, \ldots, n, \end{cases} \tag{14}$$

provided this sequence is decreasing. By (4), it suffices to show that $a_L \geq \eta B$. But this will depend on the values of λ and L, which we calculate next.

We first note that (14) implies that:

$$\sum_{k=1}^{i} P_k = \frac{1}{\lambda}\left[\frac{r_i d_i - r_{i+1} d_{i+1}}{d_i - d_{i+1}}\right],\ i = 1, \ldots, L-1. \tag{15}$$

This can be shown by straightforward induction. (Clearly, the expression on the right hand side of (15) must be increasing. This follows from (5).) In addition,

$$P_L = \frac{a_L - a_{L+1}}{d_L} = \frac{1}{d_L}\left[\frac{1}{\lambda} \frac{r_L - r_{L-1}}{\frac{1}{d_L} - \frac{1}{d_{L-1}}} - \eta B\right]. \tag{16}$$

Combining (16) with (15) for $i = L-1$ we derive the value of λ that satisfies the sum-power constraint (9):

$$\lambda = \frac{r_L d_L}{P_0 d_L + \eta B},$$

therefore we have the following expression for a_L:

$$a_L = \frac{P_0 d_L + \eta B}{r_L d_L} \times \frac{r_L - r_{L-1}}{\frac{1}{d_L} - \frac{1}{d_{L-1}}}. \tag{17}$$

We now simply define the cutoff index L to be the largest i for which the following inequality holds:

$$\frac{P_0 d_i + \eta B}{r_i d_i} \times \frac{r_i - r_{i-1}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}} \geq \eta B \Leftrightarrow \frac{P_0}{\eta B} \geq \frac{\frac{r_{i-1}}{d_i} - \frac{r_i}{d_{i-1}}}{r_i - r_{i-1}}. \tag{18}$$

Note that the expression on the far right side of (18) is increasing, as follows from (6). Therefore, (18) will hold for a contiguous range of indices i, and the largest is selected as L. Also note that the set of indices is never empty, as for $i = 1$ we arrive at the trivial identity $\frac{P_0}{\eta B} \geq 0$.

With this selection of L, $a_L \geq \eta B$ and the sequence $\{a_i\}$ defined in (14) is decreasing. Therefore, all the constraints (9)-(12) are satisfied. It remains to show that (13) is also satisfied. By using (14), we readily have that $\sum_{k=1}^{L} \frac{r_k - r_{k-1}}{a_k} = \frac{\lambda}{d_L}$. Therefore, it easily follows that (13) are equivalent to

$$\eta B > \frac{1}{\lambda} \frac{r_i - r_L}{\frac{1}{d_i} - \frac{1}{d_L}}, \quad i = L+1, \ldots, n. \tag{19}$$

Because of (7), it suffices to show that (19) holds for $i = L+1$. In addition, straightforward algebra shows that the following equivalence also holds:

$$\eta B > \frac{1}{\lambda} \frac{r_{L+1} - r_L}{\frac{1}{d_{L+1}} - \frac{1}{d_L}} \Leftrightarrow \frac{P_0}{\eta B} < \frac{\frac{r_L}{d_{L+1}} - \frac{r_{L+1}}{d_L}}{r_{L+1} - r_L}.$$

However, the second relation is satisfied, by the way we have defined L. Therefore, the inequalities (13) are also satisfied. This concludes our proof, as we have found a power distribution that satisfies all the requirements (9)-(13). We now state our result in the form of a theorem:

Theorem 2: Let L *be the largest index* $i \in \{1, \ldots, n\}$ *that satisfies the inequality*

$$\frac{P_0}{\eta B} \geq \frac{\frac{r_{i-1}}{d_i} - \frac{r_i}{d_{i-1}}}{r_i - r_{i-1}},$$

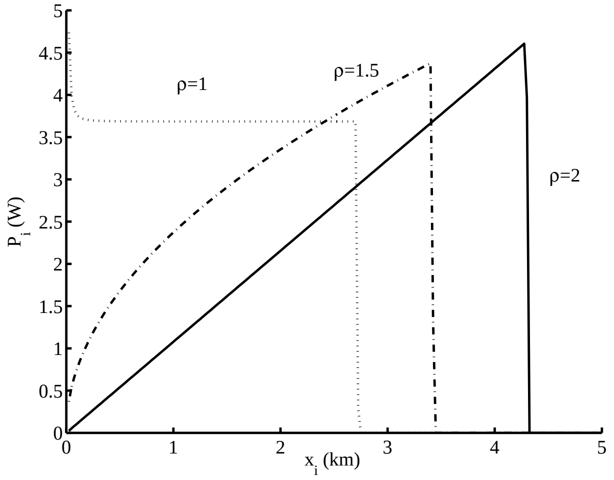

Fig. 2. The optimal power allocation in a multiple access network consisting of a receiver placed in the origin, and 200 transmitters placed uniformly along the x-axis with a separation of $25\ \mathrm{m}$ from each other, and for various values of the reward exponent ρ.

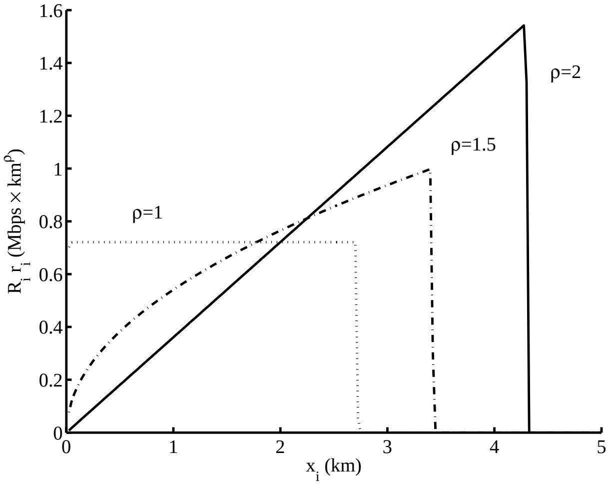

Fig. 3. The distribution of rate-reward products induced by the power distribution of Fig. 2.

where the sequence on the right hand side is increasing, by our basic assumption (4). Also let

$$a_i = \begin{cases} \lambda \cdot \frac{r_i - r_{i-1}}{\frac{1}{d_i} - \frac{1}{d_{i-1}}} & i = 1, \dots, L, \\ \eta B & i = L + 1, \end{cases}$$

where $\lambda = \frac{r_L d_L}{P_0 d_L + \eta B}$*. The maximum transport capacity is*

$$C_T^{\max} = B \sum_{i=1}^{L} (r_i - r_{i-1}) \log_2(a_i) - B r_L \log_2(\eta B),$$

and the unique power distribution that achieves it is given by:

$$P_i^* = \begin{cases} \frac{a_i - a_{i+1}}{d_i} & i = 1, \dots, L, \\ 0 & i = L + 1, \dots, n. \end{cases}$$

Our result is conceptually straightforward: The available power should be distributed among the first L transmitters, and, the greater the available power, the larger L becomes. The precise allocation of power among the first L users will depend on the exact shape of the reward and decay functions.

As a numerical example, let us consider a multiple access channel that consists of a single receiver, placed at the origin, and 200 transmitters, placed uniformly along the x-axis with a separation of $25\ \mathrm{m}$ from each other. The total power available is $P_0 = 400$ W, the available bandwidth $B = 10$ MHz, the noise spectral density is $\eta = 10^{-16}\ \frac{\mathrm{W}}{\mathrm{Hz}}$, and a monomial power decay function with $\gamma = 3$ and $K = 0.1\ \mathrm{m}^3$ is assumed. In Fig. 2 we plot the optimal distribution of powers, assuming a monomial reward function, and for the cases $\rho = 1$, $\rho = 1.5$, and $\rho = 2$. In Fig. 3, we plot the corresponding distributions of the rate-reward products of the individual transmitters. Finally, in Fig. 4 we compare the optimal power allocation for the case $\rho = 1$, with the allocation induced if the nodes lying in the intervals $(1000\ \mathrm{m}, 1400\ \mathrm{m})$ and $(4200\ \mathrm{m}, 5000\ \mathrm{m})$ are removed. As can be seen from Theorem 2, the nodes that lie directly on the borders of the 'forbidden regions' take for themselves most of the power that was allocated to the nodes that were removed. The powers allocated to the rest of the nodes also change, and in fact in the same proportion, through the change in the value of λ.

IV. OPTIMAL POWER ALLOCATION IN THE LIMIT OF AN INFINITE NUMBER OF NODES

We would like to characterize the optimal power allocation in the limit where the number of transmitters n goes to infinity, and the transmitters completely cover the whole semi-axis $\mathbf{R}^+$. Formally, we require that as $n \to \infty$, the distance of a point $x \in \mathbf{R}^+$ from its closest transmitter converges to 0.

Clearly, it no longer makes sense to discuss in terms of the powers allocated to individual nodes, since the power allocated to almost all of them will have to converge to 0, but rather in terms of the **power density function** $f(x)$, defined such that the power allocated to the transmitters lying in the interval (x, y) converges to $\int_x^y f(x)\,dx$. The power density function and the corresponding limit of the transport capacity are given by the following corollary:

Corollary 1: Let $r(\cdot)$ *and* $d(\cdot)$ *be differentiable in* $(0, \infty)$*. Let the* ***cutoff point*** x_L *be the largest* $x \in \mathbf{R}^+$ *that satisfies the inequality*

$$\frac{P_0}{\eta B} \geq g(x) \triangleq -\left.\frac{(rd)'}{r' d^2}\right|_x,$$

where $g : \mathbf{R}^+ \to \mathbf{R}$ *is the* ***cutoff function****, and assume that* $x_L < \infty$*. Also, let*

$$a(x) \triangleq \begin{cases} \frac{P_0 d(x_L) + \eta B}{r(x_L) d(x_L)} \cdot \left.\frac{r'}{(\frac{1}{d})'}\right|_x & 0 < x \leq x_L, \\ \eta B & x > x_L, \end{cases}$$

be the ***residual power function****. As the number of transmitters approaches infinity, the maximum transport capacity converges to the value*

$$C_T^{\max} = B \int_0^{x_L} r'(x) \log_2 a(x)\,dx - B r(x_L) \log_2(\eta B)$$

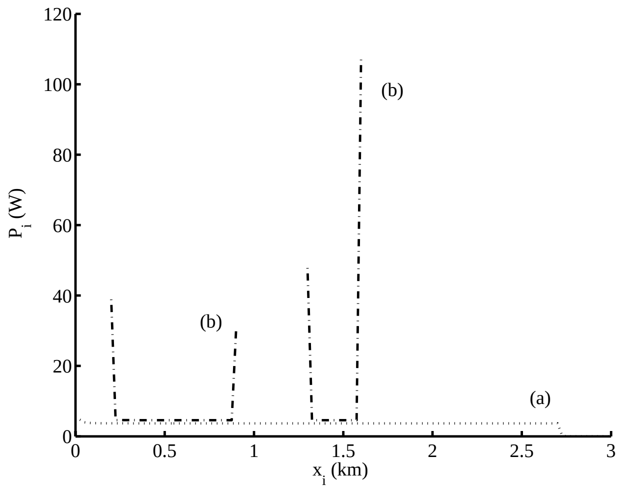

Fig. 4. **(a)** The optimal power distribution of the network of Fig. 2, for the case $\rho = 1$. **(b)** The optimal power allocation in the network of Fig. 2, for the case $\rho = 1$, and if all the nodes outside the intervals $[200\ \mathrm{m}, 900\ \mathrm{m}]$ and $[1300\ \mathrm{m}, 1600\ \mathrm{m}]$ are removed.

and the optimal distribution of powers converges to the power density function:

$$f(x) = \frac{P_0 d(x_L) + \eta B}{r(x_L) d(x_L)} \times \frac{1}{d} \left(\frac{d^2 r'}{d'} \right)' \Bigg|_x .$$

Corollary 1 can be proved by starting from the results of Theorem 2 and taking the appropriate limits. It can easily be generalized to include the case where the transmitters are placed in subsets of the positive semi-axis. It easily follows from the expressions of Theorem 2 that, on the boundaries of these subsets, the power density function may converge to a delta function.

As an illustrative example, let us consider the case of the monomial reward and decay functions. A direct application of Corollary 1 gives:

$$\begin{aligned} x_L &= \left[K \left(\frac{P_0}{\eta B} \right) \left(\frac{\rho}{\gamma - \rho} \right) \right]^{\frac{1}{\gamma}}, \\ f(x) &= \left\{ [\eta B (\gamma - \rho]^{\frac{\rho}{\gamma}} (\rho P_0)^{\frac{\gamma - \rho}{\gamma}} K^{-\frac{\rho}{\gamma}} \right\} x^{\rho - 1}, \\ C_T^{\max} &= \frac{B}{\log_2 e} \left[\frac{K P_0}{\eta B} \right]^{\frac{\rho}{\gamma}} \left(\frac{\gamma}{\rho} - 1 \right)^{1 - \frac{\rho}{\gamma}}. \end{aligned} \tag{20}$$

It is interesting to compare the transport capacity with the rate-reward product C_T^{opt} of a single transmitter-receiver pair, separated by a distance x_{opt} that maximizes it. C_T^{opt} is given by (11) of [5]. A comparison of that equation with (20) shows that the quotient $C_T^{\max}/C_T^{\mathrm{opt}}$ is only a function of $\frac{\rho}{\gamma}$, i.e., $C_T^{\max}/C_T^{\mathrm{opt}} = H(\frac{\rho}{\gamma})$. As shown in Fig. 5, $H(x)$ is a strictly decreasing, convex function with $\lim_{x \to 0^+} H(x) = e$ and $\lim_{x \to 1^-} H(x) = 1$.

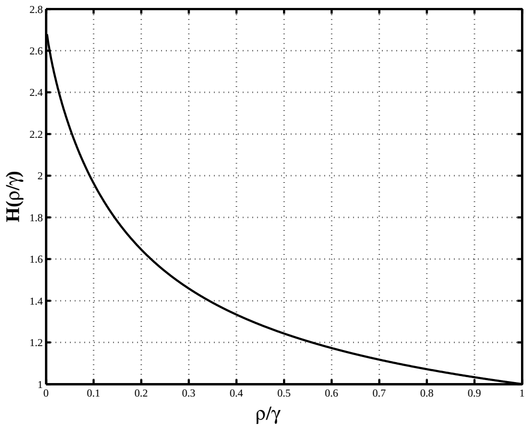

Fig. 5. The function $H(\frac{\rho}{\gamma})$, that represents the gains of using multiple transmitters over using a single transmitter, placed at the optimal distance x_{opt}, in the case of monomial decay and reward functions.

V. THE GAUSSIAN BROADCAST CHANNEL

A. Network Model and Problem Formulation

Let us consider the Gaussian broadcast channel of Fig. 6, that consists of a transmitter T with total power P_0, and n receivers $R_1, R_2, \ldots, R_n$, placed at increasing distances from the transmitter $0 < x_1 < x_2 < \ldots < x_n$. Each receiver R_i is susceptible to additive white Gaussian noise of spectral density η_i, and the total bandwidth available for communication is B. As with the multiple access channel of Fig. 1, we assume that if the transmitter sends a signal with power p, the signal will arrive at receiver R_i with power $p \times d(x_i)$, where $d(\cdot)$ is the decay function.

The capacity region of this network, i.e., the set of all combinations of rates with which the transmitter can simultaneously send information to the receivers, is known [4]. In particular, a set of rates $\{R_i\}$ is achievable iff there is a vector of powers $\mathbf{P} \triangleq (P_1, P_2, \ldots, P_n)$, such that $P_i \geq 0$ and $\sum_{i=1}^{n} P_i \leq P_0$, for which the following inequalities are satisfied:

$$R_i \leq B \log_2 (1 + \frac{d_i P_i}{\eta_i B + d_i \sum_{j=1}^{i-1} P_j}),\ \forall i = 1, \ldots, n.$$

To achieve any point in the capacity region, the transmitter encodes the message intended for receiver R_i independently of the others, and transmits it with power P_i, simultaneously with the signals intended for all other receivers. Receiver R_i decodes first all the signals intended for nodes $R_{i+1}, \ldots, R_n$, and then its own. The signals intended for nodes $R_1, \ldots, R_{i-1}$ appear to R_i as thermal noise.

As with the multiple access channel, we define the transport capacity as:

$$C_T(\{R_i\}) \triangleq \sum_{i=1}^{n} r_i R_i,$$

where $r_i \triangleq r(x_i)$, and $r(\cdot)$ is the reward function.

We would like to determine the power allocation that maximizes the transport capacity, i.e., we would like to solve

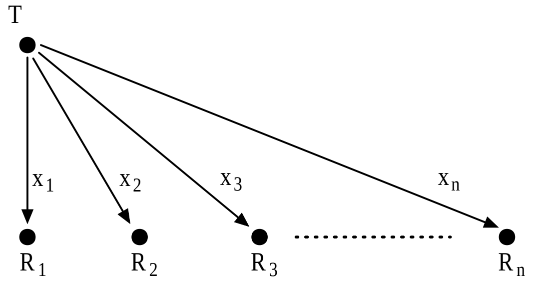

Fig. 6. The Gaussian broadcast channel, which consists of a single receiver T and n receivers R_i, with $i = 1, 2, \ldots, n$, placed at increasing distances x_i from the transmitter.

the following optimization problem:

$$\begin{array}{ll} \text{maximize:} & B\sum_{i=1}^{n} r_i \log_2\left(1+\frac{d_iP_i}{\eta_i B+d_i\sum_{j=1}^{i-1}P_j}\right), \\ \text{subject to:} & \begin{cases} \sum_{j=1}^{n} P_j \le P_0, \\ P_i \ge 0,\ i=1,\ldots,n. \end{cases} \end{array}$$

Defining $s_i \triangleq \frac{\eta_i B}{d_i}$, we rewrite the optimization problem as:

$$\begin{array}{ll} \text{minimize:} & f_0(\mathbf{P}) \triangleq -B\sum_{i=1}^{n} r_i \log_2\left(\frac{s_i+\sum_{j=1}^{i}P_j}{s_i+\sum_{j=1}^{i-1}P_j}\right), \\ \text{subject to:} & \begin{cases} g_i(\mathbf{P}) \triangleq -P_i \le 0,\ i=1,\ldots,n, \\ g_{n+1}(\mathbf{P}) \triangleq \sum_{j=1}^{n} P_j - P_0 \le 0. \end{cases} \end{array}$$

We note that the domain of the optimization problem, i.e. the subset of $\mathbf{R}^n$ where all the constraints are satisfied, is compact, and that the objective function $f_0(\cdot)$ is continuous within this set. Therefore the infimum of $f_0(\cdot)$ in the constraint set is achieved, and it makes sense to discuss about a minimum.

Let $f_0(\cdot)$ be minimized at a point $\mathbf{P}$, and let $I(\mathbf{P})$ be the set:

$$I(\mathbf{P}) \triangleq \{i \in \{1,\ldots,n+1\} : g_i(\mathbf{P}) = 0\},$$

i.e., the set of indices of all constraints which are 'active' at $\mathbf{P}$. The function $f(\cdot)$ is differentiable in the domain, the functions $g_i(\cdot)$ are differentiable and convex, and there is a point $\mathbf{x} \in \mathbf{R}^n$ such that $g_i(\mathbf{x}) < 0$ for all $i = 1, \ldots, n+1$. Therefore, it follows from Lemma 5.9 and Theorem 5.8 of [9] that there are multipliers $u_i \ge 0$, for $i \in I(\mathbf{P})$ such that the following Karush-Kuhn-Tucker (KKT) condition is satisfied:

$$\nabla f_0(\mathbf{P}) + \sum_{i \in I(\mathbf{P})} u_i \nabla g_i(\mathbf{P}) = \mathbf{0}.$$

Note that $n+1 \in I(\mathbf{P})$. Indeed, if the last constraint is not satisfied with equality, there is extra power left that could be allocated to the last receiver, thus increasing its rate without adversely affecting the rates of anyone else.

By substituting for the $\nabla g_i(\mathbf{P})$, the KKT condition can be written as follows:

$$\frac{\partial f_0}{\partial P_i} - u_i + u_{n+1} = 0,\ i = 1, \ldots, n,$$

where in addition the $u_i \ge 0$, $i = 1, \ldots, n$ must satisfy the constraints:

$$u_i P_i = 0,\ i = 1, \ldots, n.$$

Note that, if $i \notin I(\mathbf{P})$, $P_i > 0$, and so $u_i = 0$.

To conclude, and setting $\nu \triangleq u_{n+1}$, the following equations must be satisfied for any power vector $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ that maximizes the transport capacity:

$$\begin{aligned} \sum_{j=1}^{n} P_j &= P_0, && (21) \\ P_i &\ge 0,\ i=1,\ldots,n, && (22) \\ \frac{\partial f_0}{\partial P_i} + \nu &\ge 0,\ i=1,\ldots,n, && (23) \\ \left[\frac{\partial f_0}{\partial P_i} + \nu\right] P_i &= 0,\ i=1,\ldots,n, && (24) \end{aligned}$$

where $\nu \ge 0$.

Straightforward calculation shows that:

$$\frac{\partial f_0}{\partial P_i} = \sum_{j=i+1}^{n} \frac{r_j P_j}{(s_j+\beta_j)(s_j+\beta_{j-1})} - \frac{r_i}{s_i+\beta_i}, \tag{25}$$

where we defined:

$$\beta_i \triangleq \sum_{j=1}^{i} P_j.$$

As β_i represents the power that has been allocated to all nodes up to node i, we will refer to $\{\beta_i\}$ as the **cumulative power sequence**.

B. A Basic Assumption

To derive the optimal power distribution, we will have to make the following assumption:

$$\frac{s_{i+1}-s_i}{s_i-s_{i-1}} > \frac{r_{i+1}-r_i}{r_i-r_{i-1}},\ \forall i = 1, \ldots, n-1, \tag{26}$$

where we define $r_0 \triangleq 0$, $s_0 \triangleq 0$. Note that, since $s_i \triangleq \frac{\eta_i B}{d_i}$, in the special case when $\eta_1 = \eta_2 = \ldots = \eta_n$, (26) is identical to (4), with the exception that we now have a strict inequality. Furthermore, the intuition is the same: when (26) holds, the deterioration of the channel gain with distance is faster than the increase of the reward with distance. Also note that exactly the same assumption was also used in [3].

Lemma 2:

(i) *Equations (26) are equivalent with:*

$$\frac{s_i r_{i-1} - r_i s_{i-1}}{r_i - r_{i-1}} < \frac{s_{i+1} r_i - r_{i+1} s_i}{r_{i+1} - r_i},\ i = 1, \ldots, n.$$

(ii) *Equations (26) imply that:*

$$\frac{r_i}{s_i} > \frac{r_{i+1}}{s_{i+1}},\ \forall i = 1, \ldots, n-1.$$

(iii) *Equations (26) imply that* $\forall i = 1, \ldots, n$, $\forall k = 1, \ldots, n-i$*:*

$$\begin{aligned} \frac{s_{i+k}-s_{i-1}}{s_i-s_{i-1}} &> \frac{r_{i+k}-r_{i-1}}{r_i-r_{i-1}}, && (27) \\ \frac{s_{i+k}r_{i-1}-s_{i-1}r_{i+k}}{r_{i+k}-r_{i-1}} &> \frac{s_i r_{i-1}-s_{i-1}r_i}{r_i-r_{i-1}}. && (28) \end{aligned}$$

Proof: (i) To prove the equivalence of the two inequalities, we cross multiply both of them, and note that we arrive at identical inequalities.

(ii) We use induction. First, we note that, for $i = 1$, (26) gives $\frac{s_2-s_1}{s_1} > \frac{r_2-r_1}{r_1}$. By cross-multiplying, we arrive at $\frac{r_1}{s_1} > \frac{r_2}{s_2}$. Next, let us suppose that the inequality holds for $i = k < n-1$: $\frac{r_k}{s_k} > \frac{r_{k+1}}{s_{k+1}}$. We note that:

$$\frac{s_{k+2}r_{k+1} - r_{k+2}s_{k+1}}{r_{k+2} - r_{k+1}} > \frac{s_{k+1}r_k - r_{k+1}s_k}{r_{k+1} - r_k} > 0.$$

The first inequality comes from part (i) for $i = k + 1$. The second inequality comes from the inductive hypothesis. Therefore, it follows that the left hand side must be strictly positive, and as $r_{k+2} > r_{k+1}$, we must have $s_{k+2}r_{k+1} - r_{k+2}s_{k+1} > 0$. Therefore, the inequality also holds for $k + 1$.

(iii) The proof of (27) is identical in structure to the proof of (7), with the only difference being that we substitute inequalities with strict inequalities, and $\frac{1}{d_i}$ with s_i. Equation (28) follows by noting that if we cross-multiply both (27) and (28) we arrive at the same inequality. □

C. The Closed Form Solution

In this section we derive the form of the optimal power allocation $\mathbf{P} = (P_1, \ldots, P_n)$, using the fact that it must satisfy (21)-(24). The proof will be given as a sequence of lemmas.

Lemma 3: $P_1 > 0$.

Proof: We will assume that $P_1 = 0$, and arrive at a contradiction. In particular, let us assume that $P_1 = P_2 = \ldots = P_l = 0$ and $P_{l+1} > 0$, for some $l \geq 1$. Equations (23) and (25) for $i = 1$ give (noting that $\beta_1 = 0$):

$$\sum_{j=l+1}^{n} \frac{r_j P_j}{(s_j + \beta_j)(s_j + \beta_{j-1})} - \frac{r_1}{s_1} + \nu \geq 0. \qquad (29)$$

Equations (24) and (25) for $i = l + 1$ give:

$$\sum_{j=l+2}^{n} \frac{r_j P_j}{(s_j + \beta_j)(s_j + \beta_{j-1})} - \frac{r_{l+1}}{s_{l+1} + P_{l+1}} + \nu = 0. \qquad (30)$$

Subtracting (30) from (29), and simplifying, we arrive at $\frac{r_{l+1}}{s_{l+1}} \geq \frac{r_1}{s_1}$, which is not possible by Lemma 2(ii). □

Lemma 4: There are no gaps in the set of receivers that are allocated non-zero power, i.e., there are no l, m, such that:

$$P_l > 0,\ P_{l+1} = \ldots = P_{l+m} = 0,\ P_{l+m+1} > 0. \qquad (31)$$

Proof: We will assume that there are l, m, for which (31) hold, and will arrive at a contradiction. Equations (24) and (25), applied for l and $l + m + 1$, give:

$$\sum_{j=l+m+1}^{n} \frac{r_j P_j}{(s_j + \beta_j)(s_j + \beta_{j-1})} + \nu = \frac{r_l}{s_l + \beta_l}, \qquad (32)$$

$$\sum_{j=l+m+2}^{n} \frac{r_j P_j}{(s_j + \beta_j)(s_j + \beta_{j-1})} + \nu = \frac{r_{l+m+1}}{s_{l+m+1} + \beta_{l+m+1}}. \qquad (33)$$

Subtracting (33) from (32), and noting that $\beta_{l+m+1} = \beta_l + P_{l+m+1}$ and $\beta_{l+m} = \beta_l$, we arrive, after straightforward manipulations, at:

$$\beta_l = \frac{r_l s_{l+m+1} - s_l r_{l+m+1}}{r_{l+m+1} - r_l}. \qquad (34)$$

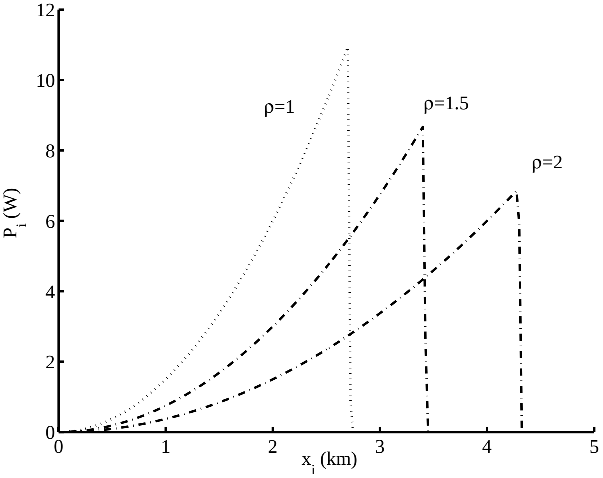

Fig. 7. The optimal power allocation in a broadcast network consisting of a transmitter placed in the origin, and 200 receivers placed uniformly along the x-axis with a separation of 25 m from each other, and for various values of the reward exponent ρ.

On the other hand, (23) and (25), applied for $i = l + 1$, give:

$$\sum_{j=l+m+1}^{n} \frac{r_j P_j}{(s_j + \beta_j)(s_j + \beta_{j-1})} + \nu \geq \frac{r_{l+1}}{s_{l+1} + \beta_l}. \qquad (35)$$

In (35), we have used the fact that $\beta_l = \beta_{l+1}$. Subtracting (32) from (35) we arrive, after straightforward manipulations, to

$$\beta_l \leq \frac{r_l s_{l+1} - r_{l+1} s_l}{r_{l+1} - r_l}.$$

This inequality, together with (34) and (28), lead to a contradiction. □

From Lemmas 3 and 4, it follows that there is a **cutoff index** L, such that $P_i > 0$ for all $i = 1, \ldots, L$, and $P_i = 0$ for all $i = L+1, \ldots, n$. Using this knowledge, we can derive the value of the β_i:

Lemma 5:

$$\beta_i = \begin{cases} \frac{r_i s_{i+1} - r_{i+1} s_i}{r_{i+1} - r_i} & i = 1, \ldots, L-1, \\ P_0, & i = L, \ldots, n. \end{cases}$$

Proof: For $i \geq L$, the result follows immediately by the definition of the β_i. If $i < L$, then as $P_i, P_{i+1} > 0$, it follows from (24) and (25) that:

$$\begin{aligned} \sum_{j=i+1}^{L} \frac{r_j P_j}{(s_j + \beta_j)(s_j + \beta_{j-1})} + \nu &= \frac{r_i}{s_i + \beta_i}, \\ \sum_{j=i+2}^{L} \frac{r_j P_j}{(s_j + \beta_j)(s_j + \beta_{j-1})} + \nu &= \frac{r_{i+1}}{s_{i+1} + \beta_{i+1}}. \end{aligned}$$

Subtracting the second equation from the first, and after performing some straightforward manipulation, we arrive at the result. □

As the power allocation can be readily derived from the sequence $\{\beta_i\}$, the only thing we are missing to conclude the

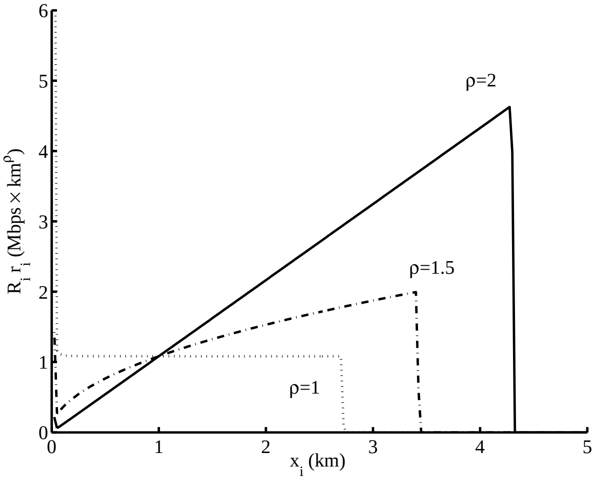

Fig. 8. The distribution of rate-reward products induced by the power distribution of Fig. 7.

proof is the exact value of L. For this, we first observe that, as $P_L > 0$, it follows from (24) and (25) that:

$$\nu = \frac{r_L}{s_L + P_0}.$$

It is then straightforward to show that for the case $i = L + 1, \dots, n$, (23) become equivalent to:

$$P_0 \leq \frac{r_L s_i - s_L r_i}{r_i - r_L}. \tag{36}$$

By Lemma 2(iii), if (36) is satisfied for $i = L + 1$, it will be satisfied for all $i > L + 1$. However, in order to satisfy (36) for $i = L+1$, and at the same time ensure that β_{L-1}, as given by Lemma 5, is smaller than P_0, L can only be set as follows:

$$L = \max\{i : \frac{r_{i-1}s_i - r_i s_{i-1}}{r_i - r_{i-1}} < P_0\}.$$

This concludes the proof, and we are ready to state the result in the form of a theorem:

Theorem 3: Let $L = \max\{i : \frac{r_{i-1}s_i - r_i s_{i-1}}{r_i - r_{i-1}} < P_0\}$*, and*

$$\beta_i = \begin{cases} \frac{r_i s_{i+1} - r_{i+1} s_i}{r_{i+1} - r_i} & i = 0, \dots, L-1, \\ P_0, & i = L, \dots, n. \end{cases}$$

The maximum transport capacity is:

$$C_T^{\max} = B \sum_{i=1}^{L} r_i \log_2 \left(\frac{s_i + \beta_i}{s_i + \beta_{i-1}} \right),$$

and the optimal power allocation that achieves it is:

$$P_i = \begin{cases} \beta_i - \beta_{i-1}, & i = 1, \dots, L, \\ 0, & i = L+1, \dots, n. \end{cases}$$

We must emphasize that a closed form for the optimal power allocation, also assuming (26), and compatible to that of Theorem 3, appeared first in [3]. There, however, it was shown that the transmitter allocates positive power to a contiguous group of receivers, and perhaps one or two extra, outlying receivers. There may be inactive receivers (i.e. receivers that receive no power) separating the contiguous group from the outlying receivers. Our work shows that in fact only a much less general scenario can occur, i.e., the first L receivers will receiver all the power, for some L, and the rest will receive no power at all. In addition, our derivations are shorter, and use a standard optimization tool, i.e., the KKT conditions.

As a numerical example, let us consider a broadcast channel that consists of a single transmitter, placed at the origin, and 200 receivers, placed uniformly along the x-axis with a separation of 25 m from each other. The total power available at the transmitter is $P_0 = 400$ W, the available bandwidth $B = 10$ MHz, the noise spectral density of all receivers is $\eta = 10^{-16}\ \frac{\mathrm{W}}{\mathrm{Hz}}$, and a monomial power decay function with $\gamma = 3$ and $K = 0.1\ \mathrm{m}^3$ is assumed. In Fig. 7 we plot the optimal distribution of powers, assuming a monomial reward function, and for the cases $\rho = 1$, $\rho = 1.5$, and $\rho = 2$. In Fig. 3, we plot the corresponding distributions of the rate-reward products of the individual transmitters. Finally, in Fig. 4 we compare the optimal power allocation for the case $\rho = 1$, with the allocation induced if the nodes lying outside the intervals $(600\ \mathrm{m}, 900\ \mathrm{m})$ and $(1300\ \mathrm{m}, 2650\ \mathrm{m})$ are removed. As can be seen from Theorem 3, the nodes that lie directly on the borders of the 'forbidden regions' take for themselves *all* the power that was allocated to the nodes that were removed. Contrary to the multiple access case, the powers allocated to the rest of the nodes do not change at all.

D. Optimal Power Allocation in the Limit of an Infinite Number of Nodes

As we did in Section IV for the multiple access channel, let us now consider the case where we place a very large number of receivers on the whole positive semi-axis $\mathbf{R}^+$. The following corollary readily follows, by using Theorem 3 and appropriately taking limits:

Corollary 2: Let $r(\cdot)$ *and* $s(\cdot)$ *be differentiable in* $(0, \infty)$*. Let*

$$\beta(x) \triangleq \frac{s'(x)r(x) - r'(x)s(x)}{r'(x)}$$

be the ***cumulative power function****, and let the* ***cutoff point*** $x_L \triangleq \sup\{x : \beta(x) < P_0\}$*. If* $x_L < \infty$*, then, as the number of receivers approaches infinity, the optimal power allocation approaches the power density function:*

$$f_{\mathrm{opt}}(x) = \begin{cases} \beta'(x) & x \leq x_L, \\ 0 & x > x_L 0. \end{cases}$$

and the optimal transport capacity converges to the value:

$$C_T^{\max} = \frac{B}{\log_2 e} \int \frac{r(x) f_{\mathrm{opt}}(x)}{s(x) + \beta(x)}\, dx.$$

As a special case, let us assume that the decay and reward functions are the monomial, and in addition all receiver face

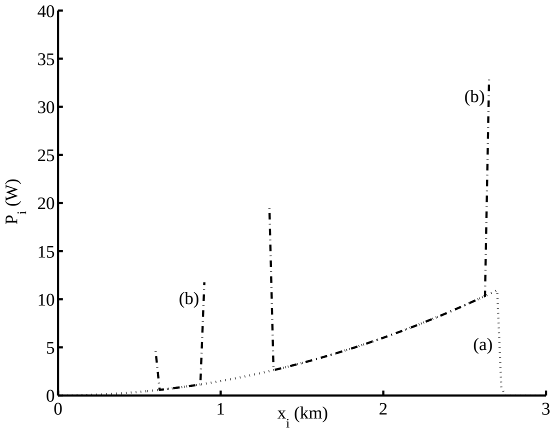

Fig. 9. **(a)** The optimal power distribution of the network of Fig. 7, for the case $\rho = 1$. **(b)** The optimal power allocation in the network of Fig. 7, for the case $\rho = 1$, and if all the nodes outside the intervals $[300\ \mathrm{m}, 900\ \mathrm{m}]$ and $[1300\ \mathrm{m}, 2650\ \mathrm{m}]$ are removed.

the same level of thermal noise, i.e., $\eta_1 = \ldots = \eta_n = \eta$. Direct application of Corollary 2 gives:

$$
\begin{aligned}
x_L &= \left[K \left(\frac{P_0}{\eta B} \right) \left(\frac{\rho}{\gamma - \rho} \right) \right]^{\frac{1}{\gamma}}, \\
f(x) &= \frac{\eta B (\gamma - \rho)\gamma}{K \rho} x^{\gamma - 1}, \\
C_T^{\max} &= \frac{B}{\log_2 e} \left[\frac{K P_0}{\eta B} \right]^{\frac{\rho}{\gamma}} \left(\frac{\gamma}{\rho} - 1 \right)^{1 - \frac{\rho}{\gamma}}.
\end{aligned}
$$

It is interesting to note that, in Section IV, we derived exactly the same formulas for the cutoff point and the transport capacity of the *multiple access* channel. There, however, the dependence of the optimal power distribution on x was of the form $x^{\rho-1}$. Therefore, the multiple access and broadcast channels seem to share important similarities, but also critical differences, that might be of interest to network designers. As the optimal transport capacities are given by identical formulas for both the multiple access and the broadcast channel, Fig. 5 also gives the gains, in terms of transport capacity, of using multiple receivers instead of a single one placed at the optimal distance.

VI. CONCLUSIONS

In this work we evaluate the traffic carrying capabilities of a Gaussian multiple access channel and a Gaussian broadcast channel. Our figure of merit is the transport capacity, a relatively new concept that is remarkably amenable to analysis and very intuitive in the context of wireless networks, but with a few exceptions [1], [2], [3], [5], [10] remains largely unexplored.

In the case of the multiple access channel, which consists of a single receiver and many transmitters, we present closed form expressions for the optimal distribution of transmitter powers under a sum-power constraint. Under a general assumption on the relation of the reward and decay functions, we find that the best strategy is that only the transmitters up to a certain distance from the receiver transmit, and the rest remain silent. We also present closed form expressions for the case when the number of nodes is very large, and the nodes cover the whole positive semi-axis.

In the case of the broadcast channel, which consists of a single transmitter and multiple receivers, we derive in closed form the optimal allocation of the transmitter power to the signals intended for the different receivers, that maximizes the transport capacity. Although this result has already been reported in the literature [3], our derivation is based on standard tools (the KKT conditions), is shorter, and leads to a simpler closed form. We also derive expressions for the optimal power allocation and associated transport capacity when there is a very large number of receivers, covering the whole positive semi-axis.

Our results allow us to quantify the gains of using successive interference cancellation, and also gain intuition about the allocations of power that promote the efficient use of the network resources in many networks that are approximated by the multiple access and broadcast channels; for example the uplink and downlink channel of cellular networks, and sensor networks that consist of a large number of sensors that must communicate with a single central station. Note that, contrary to multipoint to multipoint channels, the MAC and broadcast channels have natural bottlenecks: the unique receiver and the unique transmitter respectively. Therefore, optimizing the operation of the network is particularly important in these cases.

The power allocation problem is intimately related to the node placement problem, i.e., the problem of optimally placing a set of nodes to maximize the transport capacity, and this work can be used as a starting point of an investigation toward that direction. Also, in this work no routing of any type was investigated. These will be the subjects of future investigation.

REFERENCES

[1] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Trans. Inform. Theory*, vol. 46, no. 2, pp. 388–404, Mar. 2000.

[2] F. Baccelli, B. Blaszczyszyn, and P. Mühlethaler, "A spatial reuse Aloha MAC protocol for multihop wireless mobile networks," INRIA, Tech. Rep. RR-4955, Oct. 2003.

[3] A. Reznik and S. Verdú, "On the transport capacity of a broadcast Gaussian channel," *Communications in Information and Systems*, vol. 2, no. 2, pp. 183–216, Dec. 2002.

[4] T. Cover and J. Thomas, *Elements of Information Theory*, 1st ed. New York: Wiley Interscience, Aug. 1991.

[5] S. Toumpis, R. Müller, and J. Sayir, "On the transport capacity of a multiple access gaussian channel," in *Proc. IEEE Inter. Workshop on Wireless Ad Hoc Networks*, Oulu, Finland, May-June 2004.

[6] D. N. C. Tse and S. V. Hanly, "Multiaccess fading channels — part I: Polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Trans. Inform. Theory*, vol. 44, no. 7, pp. 2796–2816, Nov. 1998.

[7] S. V. Hanly and D. N. C. Tse, "Multiaccess fading channels — part II: Delay-limited capacities," *IEEE Trans. Inform. Theory*, vol. 44, no. 7, pp. 2816–2831, Nov. 1998.

[8] S. Boyd and L. Vandenberghe, *Convex Optimization*, 1st ed. Cambridge University Press, 2004.

[9] J. Jahn, *Introduction to the Theory of Nonlinear Optimization*. Springer-Verlag, 1994.

[10] M. Johansson and L. Xiao, "Cross-layer optimization of wireless networks using nonlinear column generation," Department of Signals, Sensors and Systems, KTH, Tech. Rep. IT-S3-REG-0302, Nov. 2003.